Integration 13G

1 a A, B are given by $6 = x^2 + 2$

$x^2 = 4$

$x = \pm 2$

So A is (−2, 6) and B is (2, 6).

b $\text{Area} = \int_{-2}^{2}\left(6-\left(x^2+2\right)\right)\,dx$

$= \int_{-2}^{2}\left(4-x^2\right)\,dx$

$= \left(4x - \frac{x^3}{3}\right)_{-2}^{2}$

$= \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right)$

$= 16 - 2\times\frac{8}{3}$

$= 10\frac{2}{3}$

2 a A, B are given by $3 = 4x - x^2$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 1, 3$

So A is (1, 3) and B is (3, 3).

b $\text{Area} = \int_{1}^{3}\left(\left(4x-x^2\right)-3\right)\,dx$

$= \int_{1}^{3}\left(4x-x^2-3\right)\,dx$

$= \left(2x^2 - \frac{x^3}{3} - 3x\right)_{1}^{3}$

$= (18-9-9)-\left(2-\frac{1}{3}-3\right)$

$= 1\frac{1}{3}$

3 $\text{Area} = \int_{-1}^{1}(\text{curve}-\text{line})\,dx$

$= \int_{-1}^{1}\left(9-3x-5x^2-x^3-(4-4x)\right)\,dx$

$= \int_{-1}^{1}\left(5+x-5x^2-x^3\right)\,dx$

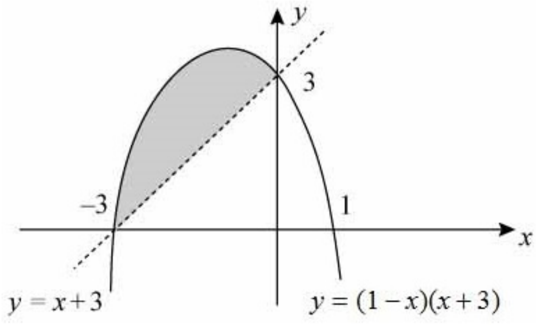
$= \left(5x+\frac{x^2}{2}-\frac{5}{3}x^3-\frac{x^4}{4}\right)_{-1}^{1}$

$= \left(5+\frac{1}{2}-\frac{5}{3}-\frac{1}{4}\right)-\left(-5+\frac{1}{2}+\frac{5}{3}-\frac{1}{4}\right)$

$= 10 - \frac{10}{3}$

$= \frac{20}{3}$ or $6\frac{2}{3}$

4 $y = (1-x)(x+3)$ is ∩ shaped and crosses the x-axis at (1, 0) and (−3, 0).
$y = x + 3$ is a straight line passing through (−3, 0) and (0, 3)



Intersections occur when

$x+3 = (1-x)(x+3)$

$0 = (x+3)(1-x-1)$

$0 = -x(x+3)$

$x = -3$ or $x = 0$

$\text{Area} = \int_{-3}^{0}\left((1-x)(x+3)-(x+3)\right)\,dx$

$= \int_{-3}^{0}\left(-x^2-3x\right)\,dx$

$= \left(-\frac{x^3}{3}-\frac{3x^2}{2}\right)_{-3}^{0}$

$= (0)-\left(\frac{27}{3}-\frac{27}{2}\right)$

$= \frac{27}{6}$

$= \frac{9}{2}$ or $4\frac{1}{2}$

5 a A is given by $x(4+x) = 12$

$x^2+4x-12 = 0$

$(x+6)(x-2) = 0$

$x = 2$ or $x = -6$

So A is (2, 12)

b R is found by $\int_{0}^{2} x(4+x)\,dx$ away from a rectangle of area 12 × 2 = 24.

So area of $R = 24 - \int_{0}^{2}\left(x^2+4x\right)\,dx$

$= 24 - \left(\frac{x^3}{3}+2x^2\right)_{0}^{2}$

5 b Area of $R = 24 - \left\{\left(\frac{8}{3}+8\right)-(0)\right\}$

$= 24 - \frac{32}{3}$

$= \frac{40}{3}$ or $13\frac{1}{3}$

6 a Intersections occur when $7 - x = x^2 + 1$

$0 = x^2 + x - 6$

$0 = (x+3)(x-2)$

$x = 2$ or -3

Area of R_1, is given by

$\int_{-3}^{2}\left(7 - x - \left(x^2+1\right)\right)\,dx$

$= \int_{-3}^{2}\left(6 - x - x^2\right)\,dx$

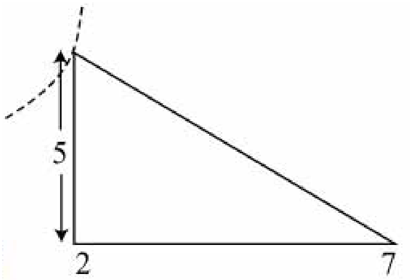
$= \left(6x - \frac{x^2}{2} - \frac{x^3}{3}\right)_{-3}^{2}$

$= \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + \frac{27}{3}\right)$

$= 20\frac{5}{6}$

b Area of R_2, is given by

$\int_{0}^{2}\left(x^2+1\right)\,dx$ + area of the triangle.



Area of $R_2 = \left(\frac{x^3}{3} + x\right)_0^2 + \frac{1}{2}\times 5\times 5$

$= \left(\frac{8}{3}+2\right) - (0) + \frac{25}{2}$

$= 17\frac{1}{6}$

7 a When $x = 1$, $y = 1 - \frac{2}{1} + 1$

$= 0$

So $(1, 0)$ lies on C.

b When $x = 8$, $y = 8^{\frac{2}{3}} - \frac{2}{8^{\frac{1}{3}}} + 1$

$= 2^2 - \frac{2}{2} + 1$

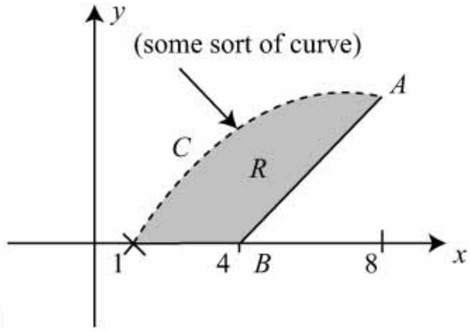
$= 4$

So $(8, 4)$ lies on C.

7 c A is the point (8, 4) and B is the point (4, 0).

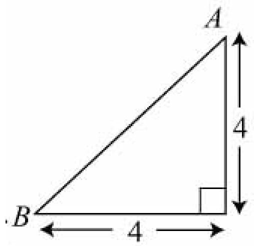
Gradient of line through AB is $\frac{4-0}{8-4} = 1$.

So the equation is $y - 0 = x - 4$ or $y = x - 4$



d Area of R is given by

$\int_{1}^{8}(\text{curve})\,dx$ − area of the triangle.



Area $R = \int_{1}^{8}\left(x^{\frac{2}{3}} - \frac{2}{x^{\frac{1}{3}}} + 1\right)dx - \frac{1}{2}\times 4\times 4$

$= \left(\frac{3}{5}x^{\frac{5}{3}} - \frac{2x^{\frac{2}{3}}}{\frac{2}{3}} + x\right)_1^8 - 8$

$= \left(\frac{3}{5}\times 32 - 3\times 4 + 8\right)$

$- \left(\frac{3}{5} - 3 + 1\right) - 8$

$= \frac{76}{5} + \frac{7}{5} - 8$

$= \frac{43}{5}$

$= 8\frac{3}{5}$

8 Area $= \int_{\frac{1}{2}}^{2}\left(\text{line } AB - \left(\frac{2}{x^2} + x\right)\right)dx$

Substitute $\frac{1}{2}$ and 2 for x into the equation to find

A is $\left(\frac{1}{2}, 8\frac{1}{2}\right)$ and B is $\left(2, 2\frac{1}{2}\right)$.

The gradient of $AB = \frac{6}{-1\frac{1}{2}} = -4$

8 So the equation is $y - 2\frac{1}{2} = -4(x-2)$

$$y = 10\frac{1}{2} - 4x$$

$$\text{Area} = \int_{\frac{1}{2}}^{2} \left(10\tfrac{1}{2} - 5x - 2x^{-2}\right) \mathrm{d}x$$

$$= \left(\frac{21}{2}x - \frac{5}{2}x^2 - \frac{2x^{-1}}{-1}\right)_{\frac{1}{2}}^{2}$$

$$= \left(\frac{21}{2}x - \frac{5}{2}x^2 + \frac{2}{x}\right)_{\frac{1}{2}}^{2}$$

$$= (21 - 10 + 1) - \left(\tfrac{21}{4} - \tfrac{5}{8} + 4\right)$$

$$= 12 - 8\tfrac{5}{8}$$

$$= 3\tfrac{3}{8} \text{ or } 3.375$$

$$= 3.38 \text{ (3 s.f.)}$$

9 a On the line, when $x = 4$, $y = 4 - \frac{1}{2} \times 4$

$$= 2$$

On the curve, when $x = 4$,

$$y = 3 \times \sqrt{4} - \sqrt{64} + 4$$

$$= 6 - 8 + 4$$

$$= 2$$

So the point (4, 2) lies on the line and the curve.

b $$\text{Area} = \int_0^4 \left(3x^{\frac{1}{2}} - x^{\frac{3}{2}} + 4 - \left(4 - \tfrac{1}{2}x\right)\right) \mathrm{d}x$$

$$= \int_0^4 \left(3x^{\frac{1}{2}} - x^{\frac{3}{2}} + \tfrac{1}{2}x\right) \mathrm{d}x$$

$$= \left(\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} - \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{4}\right)_0^4$$

$$= \left(2x^{\frac{3}{2}} - \tfrac{2}{5}x^{\frac{5}{2}} + \tfrac{1}{4}x^2\right)_0^4$$

$$= \left(2 \times 8 - \tfrac{2}{5} \times 32 + 4\right) - (0)$$

$$= 20 - \tfrac{64}{5}$$

$$= \tfrac{36}{5} \text{ or } 7.2$$

10 a $y = x^2(x+4)$

$y = 0 \Rightarrow x = 0 \text{ (twice)}, -4$

10 a Area of R_1 is

$$\int_{-4}^{0} \left(x^3 + 4x^2\right) \mathrm{d}x = \left(\frac{x^4}{4} + \frac{4}{3}x^3\right)_{-4}^{0}$$

$$= (0) - \left(\frac{4^4}{4} - \frac{4^4}{3}\right)$$

$$= \frac{4^4}{12}$$

$$= \frac{4^3}{3}$$

$$= \tfrac{64}{3} \text{ or } 21\tfrac{1}{3}$$

b Area of R_2 is $\int_0^2 \left(x^3 + 4x^2\right) \mathrm{d}x +$ area of the triangle.

$$\text{Area of } R_2 = \left(\frac{x^4}{4} + \frac{4}{3}x^3\right)_0^2 + 12(b-2)$$

$$= \left(\tfrac{16}{4} + \tfrac{32}{3}\right) - (0) + 12(b-2)$$

$$= 14\tfrac{2}{3} + 12b - 24$$

$$= -9\tfrac{1}{3} + 12b$$

Area of R_2 = area of R_1

$\Rightarrow -9\frac{1}{3} + 12b = 21\frac{1}{3}$

So $12b = 30\frac{2}{3}$

$\Rightarrow b = 2\frac{5}{9}$ or 2.56 (3 s.f.)

11 a The intersections occur when

$$10 - x = 2x^2 - 5x + 4$$

$$0 = 2x^2 - 4x - 6$$

$$0 = 2(x+1)(x-3)$$

$$x = -1 \text{ or } x = 3$$

When $x = -1$, $y = 11$, A is (–1, 11).
When $x = 3$, $y = 7$, B is (3, 7).

b $$\text{Area} = \int_{-1}^{3} [(10 - x) - (2x^2 - 5x + 4)] \, \mathrm{d}x$$

$$= \int_{-1}^{3} (10 - x - 2x^2 + 5x - 4) \, \mathrm{d}x$$

$$= \int_{-1}^{3} (6 + 4x - 2x^2) \, \mathrm{d}x$$

$$= \left[6x + 2x^2 - \tfrac{2}{3}x^3\right]_{-1}^{3}$$

$$= (18 + 18 - 18) - \left(-6 + 2 + \tfrac{2}{3}\right)$$

$$= 18 + 3\tfrac{1}{3}$$

$$= 21\tfrac{1}{3}$$

