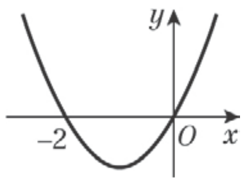
Integration 13F

1 a $y = x(x+2)$ is $\bigvee$ shaped

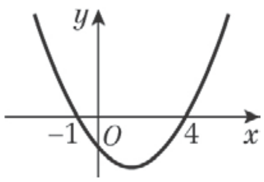
$y = 0 \Rightarrow x = 0, -2$



$$\text{Area} = \int_{-2}^{0} x(x+2)\ \mathrm{d}x$$
$$= -\int_{-2}^{0} (x^2 + 2x)\ \mathrm{d}x$$
$$= -\left(\frac{x^3}{3} + x^2\right)_{-2}^{0}$$
$$= \left\{(0) - \left(-\frac{8}{3} + 4\right)\right\}$$
$$= -\left(-\frac{4}{3}\right)$$
$$= \tfrac{4}{3} \text{ or } 1\tfrac{1}{3}$$

b $y = (x+1)(x-4)$ is $\bigvee$ shaped

$y = 0 \Rightarrow x = -1, 4$



$$\int_{-1}^{4} (x+1)(x-4)\ \mathrm{d}x = \int_{-1}^{4} (x^2 - 3x - 4)\ \mathrm{d}x$$
$$= \left(\frac{x^3}{3} - \frac{3x^2}{2} - 4x\right)_{-1}^{4}$$
$$= \left(\frac{64}{3} - \frac{3}{2} \times 16 - 16\right)$$
$$- \left(-\frac{1}{3} - \frac{3}{2} + 4\right)$$
$$= \frac{64}{3} - 40 + \frac{11}{6} - 4$$
$$= -20\tfrac{5}{6}$$

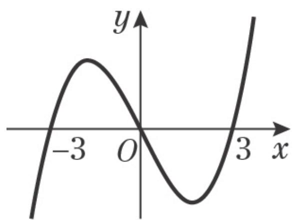
So area $= 20\tfrac{5}{6}$

c $y = (x+3)x(x-3)$

$y = 0 \Rightarrow x = -3, 0, 3$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$$\int y\mathrm{d}x = \int (x^3 - 9x)\ \mathrm{d}x$$
$$= \left(\frac{x^4}{4} - \frac{9}{2}x^2\right)$$
$$\int_{-3}^{0} y\ \mathrm{d}x = (0) - \left(\frac{81}{4} - \frac{9}{2} \times 9\right)$$
$$= +\frac{81}{4}$$
$$\int_{0}^{3} y\,\mathrm{d}x = \left(\frac{81}{4} - \frac{9}{2} \times 9\right) - (0)$$
$$= -\frac{81}{4}$$

So area $= \dfrac{81}{4} + \dfrac{81}{4}$

$= \dfrac{81}{2}$ or $40\tfrac{1}{2}$

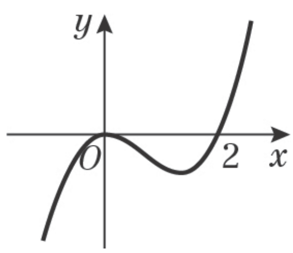
d $y = x^2(x-2)$

$y = 0 \Rightarrow x = 0\ (\text{twice}), 2$

There is a turning point at (0, 0).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



1 d Area $= -\int_0^2 x^2(x-2)\,\mathrm{d}x$

$= -\int_0^2 (x^3 - 2x^2)\,\mathrm{d}x$

$= -\left(\frac{x^4}{4} - \frac{2}{3}x^3\right)_0^2$

$= -\left\{\left(\frac{16}{4} - \frac{2}{3}\times 8\right) - (0)\right\}$

$= -\left(4 - \frac{16}{3}\right)$

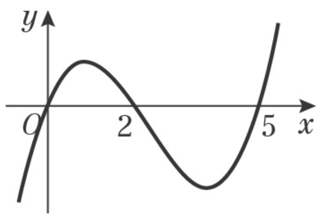
$= \frac{4}{3}$ or $1\frac{1}{3}$

e $y = x(x-2)(x-5)$

$y = 0 \Rightarrow x = 0, 2, 5$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



$\int y\,\mathrm{d}x = \int x(x^2 - 7x + 10)\,\mathrm{d}x$

$= \int (x^3 - 7x^2 + 10x)\,\mathrm{d}x$

$= \left(\frac{x^4}{4} - \frac{7}{3}x^3 + 5x^2\right)$

$\int_0^2 y\,\mathrm{d}x = \left(\frac{16}{4} - \frac{7}{3}\times 8 + 20\right) - (0)$

$= 24 - \frac{56}{3}$

$= 5\frac{1}{3}$

$\int_2^5 y\,\mathrm{d}x = \left(\frac{625}{4} - \frac{7}{3}\times 125 + 125\right) - \left(5\frac{1}{3}\right)$

$= -15\frac{3}{4}$

So area $= 5\frac{1}{3} + 15\frac{3}{4}$

$= 21\frac{1}{12}$

2 a $x(x + 3)(2 - x) = 0$

$x = 0$, $x = -3$ or $x = 2$

$A(-3, 0)$, $B(2, 0)$

b $\int_0^2 x(x+3)(2-x)\,\mathrm{d}x - \int_{-3}^0 x(x+3)(2-x)\,\mathrm{d}x$

$= \int_0^2 (-x^3 - x^2 + 6x)\,\mathrm{d}x$

$- \int_{-3}^0 (-x^3 - x^2 + 6x)\,\mathrm{d}x$

$\int_0^2 (-x^3 - x^2 + 6x)\,\mathrm{d}x$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + \frac{6x^2}{2}\right]_0^2$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_0^2$

$= \left(-\frac{2^4}{4} - \frac{2^3}{3} + 3(2)^2\right) - \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$= \left(-4 - \frac{8}{3} + 12\right)$

$= 5\frac{1}{3}$

$\int_{-3}^0 (-x^3 - x^2 + 6x)\,\mathrm{d}x$

$= \left[-\frac{x^4}{4} - \frac{x^3}{3} + 3x^2\right]_{-3}^0$

$= \left(-\frac{0^4}{4} - \frac{0^3}{3} + 3(0)^2\right)$

$- \left(-\frac{(-3)^4}{4} - \frac{(-3)^3}{3} + 3(-3)^2\right)$

$= -\left(-\frac{81}{4} + 9 + 27\right)$

$= -15\frac{3}{4}$

Total area $= 5\frac{1}{3} + 15\frac{3}{4}$

$= 21\frac{1}{12}$

3 a $f(-3) = -(-3)^3 + 4(-3)^2 + 11(-3) - 30$

$= 27 + 36 - 33 - 30 = 0$

3 b

$$\begin{array}{r|l} & -x^2+7x-10 \\ x+3 & -x^3+4x^2+11x-30 \\ & \underline{-x^3-3x^2} \\ & 7x^2+11x \\ & \underline{7x^2+21x} \\ & -10x-30 \\ & \underline{-10x-30} \\ & 0 \end{array}$$

f(x) = (x + 3)($-x^2$ + 7x − 10)

c f(x) = (x + 3)($-x$ + 2)(x − 5)

d $x = -3$, $x = 2$ or $x = 5$
(−3, 0), (2, 0) and (5, 0)

e Total area is:

$$\int_2^5 (-x^3+4x^2+11x-30)\,dx - \int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$$

$$\int_2^5 (-x^3+4x^2+11x-30)\,dx$$

$$= \left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_2^5$$

$$= \left(-\frac{5^4}{4}+\frac{4(5)^3}{3}+\frac{11(5)^2}{2}-30(5)\right) - \left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right)$$

$$= \left(-\frac{625}{4}+\frac{500}{3}+\frac{275}{2}-150\right) - \left(-4+\frac{32}{3}+22-60\right)$$

$$= 29\tfrac{1}{4}$$

$$\int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$$

$$= \left[-\frac{x^4}{4}+\frac{4x^3}{3}+\frac{11x^2}{2}-30x\right]_{-3}^2$$

$$= \left(-\frac{2^4}{4}+\frac{4(2)^3}{3}+\frac{11(2)^2}{2}-30(2)\right) - \left(-\frac{(-3)^4}{4}+\frac{4(-3)^3}{3}+\frac{11(-3)^2}{2}-30(-3)\right)$$

3 e $\int_{-3}^2 (-x^3+4x^2+11x-30)\,dx$

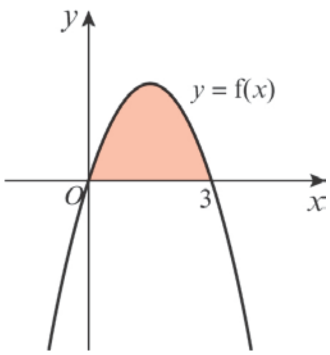
$$= \left(-4+\frac{32}{3}+22-60\right) - \left(-\frac{81}{4}-\frac{108}{3}+\frac{99}{2}+90\right)$$

$$= -114\tfrac{7}{12}$$

Total area = $29\tfrac{1}{4} + 114\tfrac{7}{12}$
= $143\tfrac{5}{6}$

Challenge

1 a $x(3 - x) = 0$
$x = 0$ or $x = 3$



f(x) = 3x − x^2

Area = $\int_0^3 (3x - x^2)\,dx$

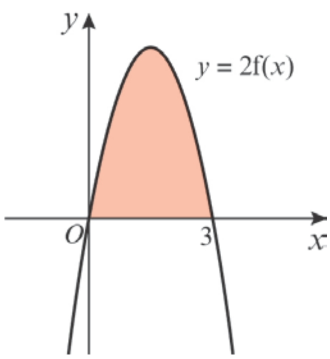
$$= \left[\frac{3x^2}{2}-\frac{x^3}{3}\right]_0^3$$

$$= \left(\frac{3(3)^2}{2}-\frac{3^3}{3}\right)-\left(\frac{3(0)^2}{2}-\frac{0^3}{3}\right)$$

$$= \left(\tfrac{27}{2}-9\right)$$

$$= 4\tfrac{1}{2}$$

b



f(x) = 6x − 2x^2

1 b Area $= \int_0^3 (6x - 2x^2)\,dx$

$$= \left[\frac{6x^2}{2} - \frac{2x^3}{3}\right]_0^3$$

$$= \left[3x^2 - \frac{2x^3}{3}\right]_0^3$$

$$= \left(3(3)^2 - \frac{2(3)^3}{3}\right) - \left(3(0)^2 - \frac{2(0)^3}{3}\right)$$

$$= (27 - 18)$$

$$= 9$$

c $f(x) = a(3x - x^2)$

Area $= a \times$ area of $f(x)$

$= a \times 4\frac{1}{2}$

$$= \frac{9a}{2}$$

d $y = f(x + a)$ is a translation of $y = f(x)$ by $\begin{pmatrix} -a \\ 0 \end{pmatrix}$.

Therefore, the area of $y = f(x + a)$ is equal to the area of $y = f(x)$.

The area of $y = f(x + a)$ is $4\frac{1}{2}$

e $f(ax) = 3ax - a^2x^2$

Area $= \int_0^{\frac{3}{a}} (3ax - a^2x^2)\,dx$

$$= \left[\frac{3ax^2}{2} - \frac{a^2x^3}{3}\right]_0^{\frac{3}{a}}$$

$$= \left(\frac{3a\left(\frac{3}{a}\right)^2}{2} - \frac{a^2\left(\frac{3}{a}\right)^3}{3}\right) - \left(\frac{3(0)^2}{2} - \frac{0^3}{3}\right)$$

$$= \left(\frac{27}{2a} - \frac{9}{a}\right)$$

$$= \frac{9}{2a}$$

2 a When $y = 0$, $x = -2$ or $x = 0$ or $x = 1$

$B(1, 0)$

$y = x(x^2 + x - 2)$

$= x^3 + x^2 - 2x$

2 a Area under the curve $= \int_0^1 (x^3 + x^2 - 2x)\,dx$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_0^1$$

$$= \left(\frac{1^4}{4} + \frac{1^3}{3} - 1^2\right) - \left(\frac{0^4}{4} + \frac{0^3}{3} - 0^2\right)$$

$$= \left(-\frac{5}{12}\right) - 0$$

$$= -\frac{5}{12}$$

So area $= \dfrac{5}{12}$

Area above the curve from $x = 0$ to $x = x$

$$\int_x^0 (x^3 + x^2 - 2x)\,dx = \frac{5}{12}$$

$$\left[\frac{x^4}{4} + \frac{x^3}{3} - x^2\right]_x^0 = \frac{5}{12}$$

$$\left(\frac{0^4}{4} + \frac{0^3}{3} - 0^2\right) - \left(\frac{x^4}{4} + \frac{x^3}{3} - x^2\right) = \frac{5}{12}$$

$$-\left(\frac{x^4}{4} + \frac{x^3}{3} - x^2\right) = \frac{5}{12}$$

$3x^4 + 4x^3 - 12x^2 + 5 = 0$

Using the factor theorem when $x = 1$

$3(1)^4 + 4(1)^3 - 12(1)^2 + 5 = 0$

So $(x - 1)$ is a factor

$$\begin{array}{r} 3x^3 + 7x^2 - 5x - 5 \\ x - 1 \overline{)\, 3x^4 + 4x^3 - 12x^2 + 0x + 5} \\ \underline{3x^4 - 3x^3} \qquad\qquad\qquad\quad \\ 7x^3 - 12x^2 \qquad\qquad \\ \underline{7x^3 - 7x^2} \qquad\qquad \\ -5x^2 + 0x \qquad \\ \underline{-5x^2 + 5x} \qquad \\ -5x + 5 \\ \underline{-5x + 5} \\ 0 \end{array}$$

2 a $3x^4 + 4x^3 - 12x^2 + 5$
$= (x - 1)(3x^3 + 7x^2 - 5x - 5)$

Using the factor theorem when $x = 1$
$3(1)^3 + 7(1)^2 - 5(1) - 5 = 0$

So $(x - 1)$ is a factor

$$\begin{array}{r}
3x^2 + 10x + 5 \\
x-1\,\overline{)\,3x^3 + 7x^2 - 5x - 5} \\
\underline{3x^3 - 3x^2} \qquad\qquad \\
10x^2 - 5x \qquad \\
\underline{10x^2 - 10x} \qquad \\
5x - 5 \\
\underline{5x - 5} \\
0
\end{array}$$

So $3x^4 + 4x^3 - 12x^2 + 5$
$= (x - 1)^2(3x^2 + 10x + 5)$

2 b When $(x - 1) = 0$, $x = 1$
When $3x^2 + 10x + 5 = 0$

$$x = \frac{-10 \pm \sqrt{10^2 - 4(3)(5)}}{2 \times 3}$$
$$= \frac{-10 \pm \sqrt{40}}{6}$$
$$= \frac{-10 \pm \sqrt{4 \times 10}}{6}$$
$$= \frac{-10 \pm 2\sqrt{10}}{6}$$
$$= \frac{-5 \pm \sqrt{10}}{3}$$

As $y = x^3 + x^2 - 2x$ going back to the original curve

When $x = \dfrac{-5+\sqrt{10}}{3}$

$$y = \left(\frac{-5+\sqrt{10}}{3}\right)^3 + \left(\frac{-5+\sqrt{10}}{3}\right)^2 - 2\left(\frac{-5+\sqrt{10}}{3}\right)$$
$$= \left(\frac{-275+85\sqrt{10}}{27}\right) + \left(\frac{35-10\sqrt{10}}{9}\right) - \left(\frac{-10+2\sqrt{10}}{3}\right)$$
$$= \frac{-80+37\sqrt{10}}{27}$$

$$A\left(\frac{-5+\sqrt{10}}{3}, \frac{-80+37\sqrt{10}}{27}\right)$$

The roots at 1 correspond to point B.

The root $\dfrac{-5-\sqrt{10}}{3}$ gives a point on the curve to the left of -2 below the x-axis, so cannot be A.