Integration 13C

1 a $\frac{dy}{dx} = 3x^2 + 2x$

$\Rightarrow y = \frac{3}{3}x^3 + \frac{2}{2}x^2 + c$

So $y = x^3 + x^2 + c$

$x = 2,\ y = 10 \Rightarrow 10 = 8 + 4 + c$

So $c = -2$

So the equation is $y = x^3 + x^2 - 2$

b $\frac{dy}{dx} = 4x^3 + \frac{2}{x^3} + 3$

$\Rightarrow y = \frac{4}{4}x^4 + \frac{2}{-2}x^{-2} + 3x + c$

So $y = x^4 - x^{-2} + 3x + c$

$x = 1,\ y = 4 \Rightarrow 4 = 1 - 1 + 3 + c$

So $c = 1$

So the equation is $y = x^4 - x^{-2} + 3x + 1$

or $y = x^4 - \frac{1}{x^2} + 3x + 1$

c $\frac{dy}{dx} = \sqrt{x} + \frac{1}{4}x^2$

$\Rightarrow y = \frac{x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{1}{4}\frac{x^3}{3} + c$

So $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{12}x^3 + c$

$x = 4, y = 11 \Rightarrow 11 = \frac{2}{3}\times 2^3 + \frac{1}{12}\times 4^3 + c$

So $c = 11 - \frac{16}{3} - \frac{16}{3} = \frac{1}{3}$

So the equation is $y = \frac{2}{3}x^{\frac{3}{2}} + \frac{1}{12}x^3 + \frac{1}{3}$

d $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - x$

$\Rightarrow y = 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} - \frac{1}{2}x^2 + c$

So $y = 6\sqrt{x} - \frac{1}{2}x^2 + c$

$x = 4, y = 0 \Rightarrow 0 = 6\times 2 - \frac{1}{2}\times 16 + c$

So $c = -4$

So the equation is $y = 6\sqrt{x} - \frac{1}{2}x^2 - 4$

e $\frac{dy}{dx} = (x+2)^2$

$= x^2 + 4x + 4$

$\Rightarrow y = \frac{1}{3}x^3 + 2x^2 + 4x + c$

$x = 1, y = 7 \Rightarrow 7 = \frac{1}{3} + 2 + 4 + c$

So $c = \frac{2}{3}$

So the equation is $y = \frac{1}{3}x^3 + 2x^2 + 4x + \frac{2}{3}$

f $\frac{dy}{dx} = \frac{x^2 + 3}{\sqrt{x}} = x^{\frac{3}{2}} + 3x^{-\frac{1}{2}}$

$\Rightarrow y = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + 3\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$

So $y = \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + c$

$x = 0, y = 1 \Rightarrow 1 = \frac{2}{5}\times 0 + 6\times 0 + c$

So $c = 1$

So the equation is $y = \frac{2}{5}x^{\frac{5}{2}} + 6x^{\frac{1}{2}} + 1$

2 $f'(x) = 2x^3 - \frac{1}{x^2}$

$= 2x^3 - x^{-2}$

So $f(x) = \frac{2}{4}x^4 - \frac{x^{-1}}{-1} + c = \frac{1}{2}x^4 + \frac{1}{x} + c$

But $f(1) = 2$

So $2 = \frac{1}{2} + 1 + c$

$\Rightarrow c = \frac{1}{2}$

So $f(x) = \frac{1}{2}x^4 + \frac{1}{x} + \frac{1}{2}$

3 $\frac{dy}{dx} = \frac{\sqrt{x} + 3}{x^2}$

$= x^{-\frac{3}{2}} + 3x^{-2}$

$\Rightarrow y = \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}} + 3\frac{x^{-1}}{-1} + c$

So $y = -2x^{-\frac{1}{2}} - 3x^{-1} + c$

$= -\frac{2}{\sqrt{x}} - \frac{3}{x} + c$

3 $x=9,\ y=0 \Rightarrow 0=-\frac{2}{3}-\frac{3}{9}+c$

So $c=\frac{2}{3}+\frac{1}{3}=1$

So the equation is $y=1-\dfrac{2}{\sqrt{x}}-\dfrac{3}{x}$

4 $y=\int(9x^2+4x-3)\,dx$

$=\dfrac{9x^3}{3}+\dfrac{4x^2}{2}-3x+c$

$=3x^3+2x^2-3x+c$

When $x=-1$ and $y=0$,

$0=3(-1)^3+2(-1)^2-3(-1)+c$

$-3+2+3+c=0$

$c=-2$

$f(x)=3x^3+2x^2-3x-2$

5 $y=\int(3x^{-\frac{1}{2}}-2x\sqrt{x})\,dx$

$=\int(3x^{-\frac{1}{2}}-2x^{\frac{3}{2}})\,dx$

$=\dfrac{3x^{\frac{1}{2}}}{\frac{1}{2}}-\dfrac{2x^{\frac{5}{2}}}{\frac{5}{2}}+c$

$=6x^{\frac{1}{2}}-\frac{4}{5}x^{\frac{5}{2}}+c$

When $x=4$ and $y=10$,

$10=6(4)^{\frac{1}{2}}-\frac{4}{5}(4)^{\frac{5}{2}}+c$

$12-\frac{128}{5}+c=10$

$c=\frac{118}{5}$

$y=6x^{\frac{1}{2}}-\frac{4}{5}x^{\frac{5}{2}}+\frac{118}{5}$

6 a $\dfrac{6x+5x^{\frac{3}{2}}}{\sqrt{x}}=\dfrac{6x+5x^{\frac{3}{2}}}{x^{\frac{1}{2}}}$

$=x^{-\frac{1}{2}}(6x+5x^{\frac{3}{2}})$

$=6x^{\frac{1}{2}}+5x$

$p=\frac{1}{2}$ and $q=1$

b $y=\int(6x^{\frac{1}{2}}+5x)\,dx$

$=\dfrac{6x^{\frac{3}{2}}}{\frac{3}{2}}+\dfrac{5x^2}{2}+c$

$=4x^{\frac{3}{2}}+\dfrac{5x^2}{2}+c$

6 b When $x=9$ and $y=100$,

$100=4(9)^{\frac{3}{2}}+\dfrac{5(9)^2}{2}+c$

$108+\frac{405}{2}+c=100$

$c=-\frac{421}{2}$

$y=4x^{\frac{3}{2}}+\frac{5}{2}x^2-\frac{421}{2}$

7 a $f(t)=\int(10-5t)\,dt$

$=10t-\dfrac{5t^2}{2}+c$

When $x=0$ and $y=0$,

$f(0)=10(0)-\dfrac{5(0)^2}{2}+c=0$

$c=0$

$f(t)=10t-\frac{5}{2}t^2$

b $f(3)=10(3)-\dfrac{5(3)^2}{2}$

$=7\frac{1}{2}$

8 a $f(t)=\int(-9.8t)\,dt$

$=-\dfrac{9.8t^2}{2}+c$

$=-4.9t^2+c$

When $x=0$ and $y=35$,

$f(0)=-4.9(0)^2+c=35$

$c=35$

$f(t)=-4.9t^2+35$

b $f(1.5)=-4.9(1.5)^2+35=23.975$

The height of the arrow is 23.975 m.

c $f(0)=35$

The height of the castle is 35 m.

d The arrow will hit the ground when the height is 0.

$-4.9t^2+35=0$

$t=\sqrt{\dfrac{-35}{-4.9}}=2.67$ or -2.67

The time must be positive, so 2.67 seconds.

e The assumption is that the ground is flat.

Challenge

1 a $f_2'(x) = f_1(x) = x^2$

So $f_2(x) = \frac{1}{3}x^3 + c$

The curve passes through $(0, 0)$.

so $f_2(0) = 0 \Rightarrow c = 0$

So $f_2(x) = \frac{1}{3}x^3$

$f_3'(x) = \frac{1}{3}x^3$

$f_3(x) = \frac{1}{12}x^4 + c$

But $c = 0$ since $f_3(0) = 0$.

So $f_3(x) = \frac{1}{12}x^4$

b $f_2(x) = \frac{1}{3}x^3,\ f_3(x) = \dfrac{x^4}{3\times 4}$

So the power of x is $n+1$ for $f_n(x)$.

The denominator is $3\times 4\times\ldots$ up to $n+1$.

$$f_n(x) = \frac{x^{n+1}}{3\times 4\times 5\times\ldots\times(n+1)}$$

2 $f_2'(x) = f_1(x) = 1$

$\Rightarrow f_2(x) = x + c$

But $f_2(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_2(x) = x + 1$

$f_3'(x) = f_2(x) = x + 1$

$\Rightarrow f_3(x) = \frac{1}{2}x^2 + x + c$

But $f_3(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_3(x) = \frac{1}{2}x^2 + x + 1$

$f_4'(x) = f_3(x) = \frac{1}{2}x^2 + x + 1$

$\Rightarrow f_4(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + c$

But $f_4(0) = 1 \Rightarrow 1 = 0 + c \Rightarrow c = 1$

So $f_4(x) = \frac{1}{6}x^3 + \frac{1}{2}x^2 + x + 1$