Differentiation 12I

1 a $f(x) = x^2 - 12x + 8$
$f'(x) = 2x - 12$
Putting $f'(x) = 0$
$2x - 12 = 0$
$x = 6$
$f(6) = 6^2 - 12 \times 6 + 8 = -28$
The least value of f(x) is −28.

b $f(x) = x^2 - 8x - 1$
$f'(x) = 2x - 8$
Putting $f'(x) = 0$
$2x - 8 = 0$
$x = 4$
$f(4) = 4^2 - 8 \times 4 - 1 = -17$
The least value of f(x) is −17.

c $f(x) = 5x^2 + 2x$
$f'(x) = 10x + 2$
Putting $f'(x) = 0$
$10x + 2 = 0$
$x = -\frac{2}{10} = -\frac{1}{5}$
$f\left(-\frac{1}{5}\right) = 5\left(-\frac{1}{5}\right)^2 + 2\left(-\frac{1}{5}\right) = \frac{5}{25} - \frac{2}{5} = -\frac{1}{5}$
The least value of f(x) is $-\frac{1}{5}$.

2 a $f(x) = 10 - 5x^2$
$f'(x) = -10x$
Putting $f'(x) = 0$
$-10x = 0$
$x = 0$
$f(0) = 10 - 5 \times 0^2 = 10$
The greatest value of f(x) is 10.

b $f(x) = 3 + 2x - x^2$
$f'(x) = 2 - 2x$
Putting $f'(x) = 0$
$2 - 2x = 0$
$x = 1$
$f(1) = 3 + 2 - 1 = 4$
The greatest value of f(x) is 4.

c $f(x) = (6 + x)(1 - x) = 6 - 5x - x^2$
$f'(x) = -5 - 2x$
Putting $f'(x) = 0$
$-5 - 2x = 0$

2 c $x = -\frac{5}{2}$
$f\left(-\frac{5}{2}\right) = \frac{7}{2} \times \frac{7}{2} = \frac{49}{4} = 12\frac{1}{4}$
The greatest value of f(x) is $12\frac{1}{4}$.

3 a $y = 4x^2 + 6x$
$\frac{dy}{dx} = 8x + 6$
Putting $8x + 6 = 0$
$x = -\frac{6}{8} = -\frac{3}{4}$
When $x = -\frac{3}{4}$,
$y = 4\left(-\frac{3}{4}\right)^2 + 6\left(-\frac{3}{4}\right) = \frac{9}{4} - \frac{9}{2} = -\frac{9}{4}$
So $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ is a stationary point.
$\frac{d^2y}{dx^2} = 8 > 0$
So $\left(-\frac{3}{4}, -\frac{9}{4}\right)$ is a minimum point.

b $y = 9 + x - x^2$
$\frac{dy}{dx} = 1 - 2x$
Putting $1 - 2x = 0$
$x = \frac{1}{2}$
When $x = \frac{1}{2}$,
$y = 9 + \frac{1}{2} - \left(\frac{1}{2}\right)^2$
$y = 9\frac{1}{4}$
So $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ is a stationary point.
$\frac{d^2y}{dx^2} = -2 < 0$
So $\left(\frac{1}{2}, 9\frac{1}{4}\right)$ is a maximum point.

c $y = x^3 - x^2 - x + 1$
$\frac{dy}{dx} = 3x^2 - 2x - 1$
Putting $3x^2 - 2x - 1 = 0$
$(3x + 1)(x - 1) = 0$

3 c So $x = -\frac{1}{3}$ or $x = 1$

When $x = -\frac{1}{3}$,

$y = \left(-\frac{1}{3}\right)^3 - \left(-\frac{1}{3}\right)^2 - \left(-\frac{1}{3}\right) + 1$

$= 1\frac{5}{27}$

When $x = 1$,

$y = 1^3 - 1^2 - 1 + 1$

$= 0$

So $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ and (1, 0) are stationary points.

$\frac{d^2y}{dx^2} = 6x - 2$

When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{1}{3}\right) - 2 = -4 < 0$

So $\left(-\frac{1}{3}, 1\frac{5}{27}\right)$ is a maximum point.

When $x = 1$, $\frac{d^2y}{dx^2} = 6(1) - 2 = 4 > 0$

So (1, 0) is a minimum point.

d $y = x(x^2 - 4x - 3) = x^3 - 4x^2 - 3x$

$\frac{dy}{dx} = 3x^2 - 8x - 3$

Putting $3x^2 - 8x - 3 = 0$

$(3x + 1)(x - 3) = 0$

So $x = -\frac{1}{3}$ or $x = 3$

When $x = -\frac{1}{3}$,

$y = \left(-\frac{1}{3}\right)^3 - 4\left(-\frac{1}{3}\right)^2 - 3\left(-\frac{1}{3}\right)$

$= \frac{14}{27}$

When $x = 3$,

$y = 3^3 - 4(3)^2 - 3(3)$

$= -18$

So $\left(-\frac{1}{3}, \frac{14}{27}\right)$ and (3, −18) are stationary points.

$\frac{d^2y}{dx^2} = 6x - 8$

3 d When $x = -\frac{1}{3}$, $\frac{d^2y}{dx^2} = 6\left(-\frac{1}{3}\right) - 8$

$= -10 < 0$

So $\left(-\frac{1}{3}, \frac{14}{27}\right)$ is a maximum point.

When $x = 3$, $\frac{d^2y}{dx^2} = 6(3) - 8$

$= 10 > 0$

So (3, −18) is a minimum point.

e $y = x + \frac{1}{x} = x + x^{-1}$

$\frac{dy}{dx} = 1 - x^{-2}$

Putting $1 - x^{-2} = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1$,

$y = 1 + \frac{1}{1}$

$= 2$

When $x = -1$,

$y = -1 + \frac{1}{-1}$

$= -2$

So (1, 2) and (−1, −2) are stationary points.

$\frac{d^2y}{dx^2} = 2x^{-3}$

When $x = 1$, $\frac{d^2y}{dx^2} = 2 > 0$

So (1, 2) is a minimum point.

When $x = -1$, $\frac{d^2y}{dx^2} = -2 < 0$

So (−1, −2) is a maximum point.

f $y = x^2 + \frac{54}{x} = x^2 + 54x^{-1}$

$\frac{dy}{dx} = 2x - 54x^{-2}$

Putting $2x - 54x^{-2} = 0$

$x = \frac{27}{x^2}$

$x^3 = 27$

$x = 3$

3 f When $x = 3$,

$y = 3^2 + \frac{54}{3}$

$= 27$

So (3, 27) is a stationary point.

$\frac{d^2y}{dx^2} = 2 + 108x^{-3}$

When $x = 3$, $\frac{d^2y}{dx^2} = 2 + \frac{108}{3^3} = 6 > 0$

So (3, 27) is a minimum point.

g $y = x - 3\sqrt{x} = x - 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 1 - \frac{3}{2}x^{-\frac{1}{2}}$

Putting $1 - \frac{3}{2}x^{-\frac{1}{2}} = 0$

$1 = \frac{3}{2\sqrt{x}}$

$\sqrt{x} = \frac{3}{2}$

$x = \frac{9}{4}$

When $x = \frac{9}{4}$,

$y = \frac{9}{4} - 3\sqrt{\frac{9}{4}}$

$= -\frac{9}{4}$

So $(\frac{9}{4}, -\frac{9}{4})$ is a stationary point.

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{3}{2}}$

When $x = \frac{9}{4}$, $\frac{d^2y}{dx^2} = \frac{3}{4} \times \left(\frac{9}{4}\right)^{-\frac{3}{2}}$

$= \frac{3}{4} \times \left(\frac{2}{3}\right)^3$

$= \frac{2}{9} > 0$

So $(\frac{9}{4}, -\frac{9}{4})$ is a minimum point.

h $y = x^{\frac{1}{2}}(x - 6) = x^{\frac{3}{2}} - 6x^{\frac{1}{2}}$

$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}}$

Putting $\frac{3}{2}x^{\frac{1}{2}} - 3x^{-\frac{1}{2}} = 0$

$\frac{3}{2}x^{\frac{1}{2}} = \frac{3}{x^{\frac{1}{2}}}$

$\frac{3}{2}x = 3$

$x = 2$

When $x = 2$,

$y = 2^{\frac{1}{2}}(-4)$

$= -4\sqrt{2}$

So $(2, -4\sqrt{2})$ is a stationary point.

$\frac{d^2y}{dx^2} = \frac{3}{4}x^{-\frac{1}{2}} + \frac{3}{2}x^{-\frac{3}{2}}$

When $x = 2$, $\frac{d^2y}{dx^2} = \frac{3}{4\sqrt{2}} + \frac{3}{4\sqrt{2}} > 0$

So $(2, -4\sqrt{2})$ is a minimum point.

i $y = x^4 - 12x^2$

$\frac{dy}{dx} = 4x^3 - 24x$

Putting $4x^3 - 24x = 0$

$4x(x^2 - 6) = 0$

$x = 0$ or $x = \pm\sqrt{6}$

When $x = 0$, $y = 0$

When $x = \pm\sqrt{6}$, $y = -36$

So (0, 0), $(\sqrt{6}, -36)$ and $(-\sqrt{6}, -36)$ are stationary points.

$\frac{d^2y}{dx^2} = 12x^2 - 24$

When $x = 0$, $\frac{d^2y}{dx^2} = -24 < 0$

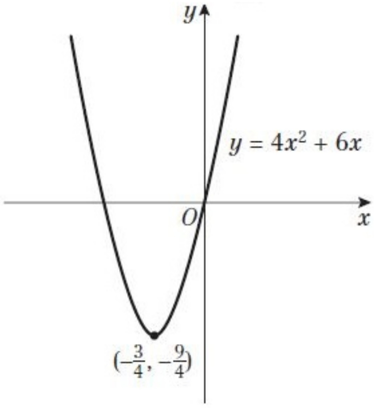
So (0, 0) is a maximum point.

When $x = \pm\sqrt{6}$,

$\frac{d^2y}{dx^2} = 12 \times 6 - 24 = 48 > 0$

So $(\sqrt{6}, -36)$ and $(-\sqrt{6}, -36)$ are minimum points.

4 a



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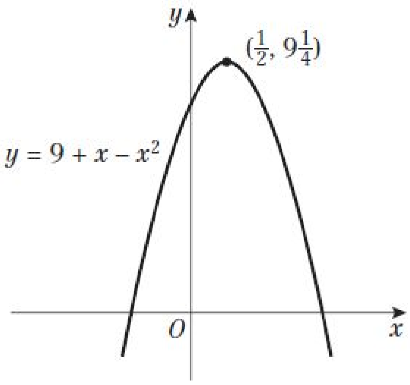

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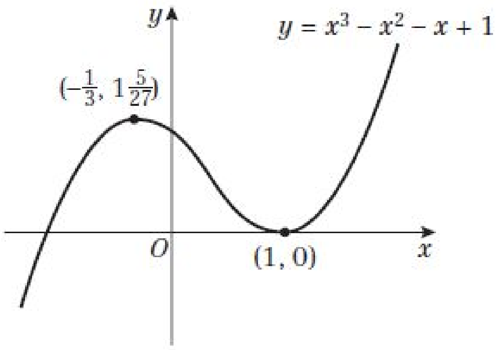

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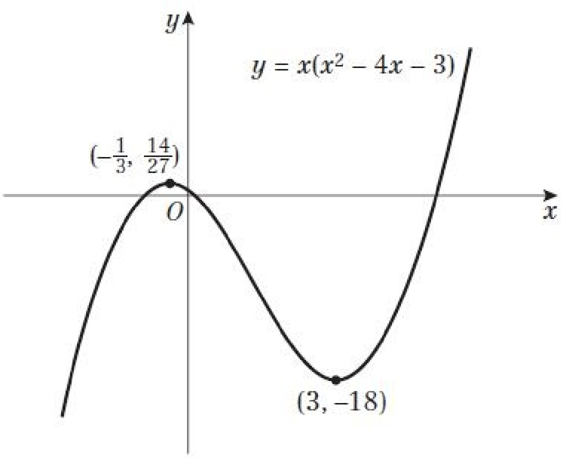


5 $y = x^3 - 3x^2 + 3x$

$\frac{dy}{dx} = 3x^2 - 6x + 3$

Putting $3x^2 - 6x + 3 = 0$

$3(x^2 - 2x + 1) = 0$

$3(x - 1)^2 = 0$

$x = 1$

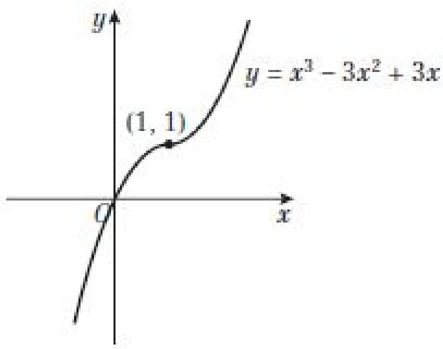
When $x = 1$, $y = 1$

So (1, 1) is a stationary point.

Considering points near to (1, 1):

x	0.9	1	1.1
$\frac{dy}{dx}$	0.03	0	0.03
	+ve	0	+ve
Shape	/	—	/

The gradient on either side of (1, 1) is positive, so (1, 1) is a point of inflection.



6 $f(x) = 27 - 2x^4$

$f'(x) = -8x^3$

Putting $-8x^3 = 0$

$x = 0$

When $x = 0$, $y = 27$

So (0, 27) is a stationary point.

$f''(x) = -24x^2$

When $x = 0$, $f''(x) = 0$, so not conclusive

Considering points near to (0, 27):

x	−0.1	0	0.1
$f'(x)$	0.008	0	−0.008
	+ve	0	−ve
Shape	/	—	\

So (0, 27) is a maximum point.

So the maximum value of f(x) is 27 and the range of values is $f(x) \leq 27$.

7 a $f(x) = x^4 + 3x^3 - 5x^2 - 3x + 1$

$f'(x) = 4x^3 + 9x^2 - 10x - 3$

Putting $4x^3 + 9x^2 - 10x - 3 = 0$

Using the factor theorem: $f'(1) = 0$,

so dividing $4x^3 + 9x^2 - 10x - 3$ by $x - 1$:

$$\begin{array}{r} 4x^2 + 13x + 3 \\ x-1\,\overline{)\,4x^3 + 9x^2 - 10x - 3} \\ \underline{4x^3 - 4x^2} \qquad\qquad\quad \\ 13x^2 - 10x \qquad \\ \underline{13x^2 - 13x} \qquad \\ 3x - 3 \\ \underline{3x - 3} \\ 0 \end{array}$$

$(x - 1)(4x^2 + 13x + 3) = 0$

$(x - 1)(4x + 1)(x + 3) = 0$

$x = 1,\ x = -\frac{1}{4}$ or $x = -3$

When $x = 1$,

$y = (1)^4 + 3(1)^3 - 5(1)^2 - 3(1) + 1$

$= -3$

When $x = -\frac{1}{4}$,

$$y = \left(-\frac{1}{4}\right)^4 + 3\left(-\frac{1}{4}\right)^3 - 5\left(-\frac{1}{4}\right)^2 - 3\left(-\frac{1}{4}\right) + 1$$

$$= \frac{357}{256}$$

When $x = -3$,

$y = (-3)^4 + 3(-3)^3 - 5(-3)^2 - 3(-3) + 1$

$= -35$

So $(1, -3)$, $(-3, -35)$ and $(-\frac{1}{4}, \frac{357}{256})$ are stationary points.

$f''(x) = 12x^2 + 18x - 10$

When $x = 1$, $f''(x) = 20 > 0$

So $(1, -3)$ is a minimum point.

When $x = -3$,

$f''(x) = 12(-3)^2 + 18(-3) - 10 = 44 > 0$

So $(-3, -35)$ is a minimum point.

When $x = -\frac{1}{4}$,

$$f''(x) = 12\left(-\frac{1}{4}\right)^2 + 18\left(-\frac{1}{4}\right) - 10$$

$$= - -\frac{55}{4} < 0$$

So $(-\frac{1}{4}, \frac{357}{256})$ is a maximum point.

7 b