Differentiation 12E

1 a Let $y = x^4 + x^{-1}$

$\frac{dy}{dx} = 4x^3 + (-1)x^{-2}$

$= 4x^3 - x^{-2}$

b Let $y = 2x^5 + 3x^{-2}$

$\frac{dy}{dx} = 5 \times 2x^{5-1} + (-2) \times 3x^{-2-1}$

$= 10x^4 - 6x^{-3}$

c Let $y = 6x^{\frac{3}{2}} + 2x^{-\frac{1}{2}} + 4$

$\frac{dy}{dx} = \frac{3}{2} \times 6x^{\frac{3}{2}-1} + \left(-\frac{1}{2}\right) \times 2x^{-\frac{1}{2}-1} + 0$

$= 9x^{\frac{1}{2}} - x^{-\frac{3}{2}}$

2 a $f(x) = x^3 - 3x + 2$

$f'(x) = 3x^2 - 3$

At $(-1, 4)$, $x = -1$

$f'(-1) = 3(-1)^2 - 3 = 0$

The gradient at $(-1, 4)$ is 0.

b $f(x) = 3x^2 + 2x^{-1}$

$f'(x) = 6x + 2(-1)x^{-2} = 6x - 2x^{-2}$

At $(2, 13)$, $x = 2$

$f'(2) = 6(2) - 2(2)^{-2} = 12 - \frac{2}{4} = 11\frac{1}{2}$

The gradient at $(2, 13)$ is $11\frac{1}{2}$.

3 a $f(x) = x^2 - 5x$

$f'(x) = 2x - 5$

When gradient is zero, $f'(x) = 0$.

$2x - 5 = 0$

$x = 2.5$

When $x = 2.5$, $y = f(2.5)$

$= (2.5)^2 - 5(2.5)$

$= -6.25$

Therefore, the gradient is zero at $(2.5, -6.25)$.

b $f(x) = x^3 - 9x^2 + 24x - 20$

$f'(x) = 3x^2 - 18x + 24$

When gradient is zero, $f'(x) = 0$.

$3x^2 - 18x + 24 = 0$

$3(x^2 - 6x + 8) = 0$

$3(x - 4)(x - 2) = 0$

$x = 4$ or $x = 2$

When $x = 4$, $y = f(4)$

$= 4^3 - 9 \times 4^2 + 24 \times 4 - 20$

$= -4$

When $x = 2$, $y = f(2)$

$= 2^3 - 9 \times 2^2 + 24 \times 2 - 20$

$= 0$

Therefore, the gradient is zero at $(4, -4)$ and $(2, 0)$.

c $f(x) = x^{\frac{3}{2}} - 6x + 1$.

$f'(x) = \frac{3}{2}x^{\frac{1}{2}} - 6$

When gradient is zero, $f'(x) = 0$.

$\frac{3}{2}x^{\frac{1}{2}} - 6 = 0$

$x^{\frac{1}{2}} = 4$

$x = 16$

When $x = 16$, $y = f(16)$

$= 16^{\frac{3}{2}} - 6 \times 16 + 1$

$= -31$

Therefore, the gradient is zero at $(16, -31)$.

d $f(x) = x^{-1} + 4x$

$f'(x) = -x^{-2} + 4$

For zero gradient, $f'(x) = 0$.

$-x^{-2} + 4 = 0$

$\frac{1}{x^2} = 4$

$x = \pm\frac{1}{2}$

When $x = \frac{1}{2}$, $y = f\left(\frac{1}{2}\right)$

$y = \left(\frac{1}{2}\right)^{-1} + 4\left(\frac{1}{2}\right)$

$= 2 + 2$

$= 4$

When $x = -\frac{1}{2}$, $y = f\left(-\frac{1}{2}\right)$

$y = \left(-\frac{1}{2}\right)^{-1} + 4\left(-\frac{1}{2}\right)$

$= -2 - 2$

Therefore, the gradient is zero at $\left(\frac{1}{2}, 4\right)$ and $\left(-\frac{1}{2}, -4\right)$

4 a Let $y = 2\sqrt{x}$

$= 2x^{\frac{1}{2}}$

$\frac{dy}{dx} = 2\left(\frac{1}{2}\right)x^{-\frac{1}{2}}$

$= x^{-\frac{1}{2}}$

$= \frac{1}{\sqrt{x}}$

b Let $y = \frac{3}{x^2}$

$= 3x^{-2}$

$\frac{dy}{dx} = 3(-2)x^{-3}$

$= -6x^{-3}$

$= -\frac{6}{x^3}$

c Let $y = \frac{1}{3x^3}$

$= \frac{1}{3}x^{-3}$

$\frac{dy}{dx} = \frac{1}{3}(-3)x^{-4}$

$= -x^{-4}$

$= -\frac{1}{x^4}$

d Let $y = \frac{1}{3}x^3(x-2)$

$= \frac{1}{3}x^4 - \frac{2}{3}x^3$

$\frac{dy}{dx} = \frac{4}{3}x^3 - \frac{2}{3}\times 3x^2$

$= \frac{4}{3}x^3 - 2x^2$

e Let $y = \frac{2}{x^3} + \sqrt{x}$

$= 2x^{-3} + x^{\frac{1}{2}}$

$\frac{dy}{dx} = -6x^{-4} + \frac{1}{2}x^{-\frac{1}{2}}$

$= -\frac{6}{x^4} + \frac{1}{2\sqrt{x}}$

f Let $y = \sqrt[3]{x} + \frac{1}{2x}$

$= x^{\frac{1}{3}} + \frac{1}{2}x^{-1}$

$\frac{dy}{dx} = \frac{1}{3}x^{-\frac{2}{3}} - \frac{1}{2}x^{-2}$

g Let $y = \frac{2x+3}{x}$

$= \frac{2x}{x} + \frac{3}{x}$

$= 2 + 3x^{-1}$

$\frac{dy}{dx} = 0 - 3x^{-2}$

$= -\frac{3}{x^2}$

h Let $y = \frac{3x^2-6}{x}$

$= \frac{3x^2}{x} - \frac{6}{x}$

$= 3x - 6x^{-1}$

$\frac{dy}{dx} = 3 + 6x^{-2}$

$= 3 + \frac{6}{x^2}$

i Let $y = \frac{2x^3+3x}{\sqrt{x}}$

$= \frac{2x^3}{x^{\frac{1}{2}}} + \frac{3x}{x^{\frac{1}{2}}}$

$= 2x^{\frac{5}{2}} + 3x^{\frac{1}{2}}$

$\frac{dy}{dx} = 5x^{\frac{3}{2}} + \frac{3}{2}x^{-\frac{1}{2}}$

j Let $y = x(x^2 - x + 2)$

$= x^3 - x^2 + 2x$

$\frac{dy}{dx} = 3x^2 - 2x + 2$

k Let $y = 3x^2(x^2 + 2x)$

$= 3x^4 + 6x^3$

$\frac{dy}{dx} = 12x^3 + 18x^2$

4 l Let $y = (3x-2)\left(4x+\frac{1}{x}\right)$

$= 12x^2 - 8x + 3 - \frac{2}{x}$

$= 12x^2 - 8x + 3 - 2x^{-1}$

$\frac{dy}{dx} = 24x - 8 + 2x^{-2}$

$= 24x - 8 + \frac{2}{x^2}$

5 a $f(x) = x(x + 1)$

$= x^2 + x$

$f'(x) = 2x + 1$

Gradient at (0, 0) = $f'(0) = 1$

b $f(x) = \frac{2x-6}{x^2}$

$= \frac{2x}{x^2} - \frac{6}{x^2}$

$= 2x^{-1} - 6x^{-2}$

$f'(x) = -2x^{-2} + 12x^{-3}$

$= -\frac{2}{x^2} + \frac{12}{x^3}$

Gradient at (3, 0) = $f'(3) = -\frac{2}{3^2} + \frac{12}{3^3}$

$= -\frac{2}{9} + \frac{12}{27}$

$= \frac{2}{9}$

c $f(x) = \frac{1}{\sqrt{x}}$

$= x^{-\frac{1}{2}}$

$f'(x) = -\frac{1}{2}x^{-\frac{3}{2}}$

Gradient at $\left(\frac{1}{4}, 2\right)$ = $f'\left(\frac{1}{4}\right) = -\frac{1}{2}\left(\frac{1}{4}\right)^{-\frac{3}{2}}$

$= -\frac{1}{2} \times 2^3$

$= -4$

5 d $f(x) = 3x - \frac{4}{x^2}$

$= 3x - 4x^{-2}$

$f'(x) = 3 + 8x^{-3}$

Gradient at (2, 5) = $f'(2) = 3 + 8(2)^{-3}$

$= 3 + \frac{8}{8} = 4$

6 $f(x) = \frac{12}{p\sqrt{x}} + x$

$= \frac{12}{p}x^{-\frac{1}{2}} + x$, $f'(2) = 3$

$f'(x) = -\frac{1}{2} \times \frac{12}{p} x^{-\frac{1}{2}-1} + 1$

$= -\frac{6}{p}x^{-\frac{3}{2}} + 1$

$f'(2) = -\frac{6}{p}(2)^{-\frac{3}{2}} + 1$

$= -\frac{6}{2p\sqrt{2}} + 1$

$-\frac{6}{2p\sqrt{2}} + 1 = 3$

$-\frac{6}{2p\sqrt{2}} = 2$

$p = -\frac{3}{2\sqrt{2}}$

$= -\frac{3}{2\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= -\frac{3}{4}\sqrt{2}$

7 a $f(x) = (2 - x)^9$

$= 2^9 + \binom{9}{1}2^8(-x) + \binom{9}{2}2^7(-x)^2 + \ldots$

$= 512 - 2304x + 4608x^2$

b $f(x) \approx 512 - 2304x + 4608x^2$

$f'(x) \approx -2304 + 2 \times 4608x^{2-1}$

$= 9216x - 2304$