Differentiation 12B

1 a $f(x) = x^2$

$$f'(2) = \lim_{h\to0}\frac{f(2+h)-f(2)}{h}$$
$$= \lim_{h\to0}\frac{(2+h)^2-2^2}{h}$$
$$= \lim_{h\to0}\frac{4+4h+h^2-4}{h}$$
$$= \lim_{h\to0}\frac{4h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(4+h)}{h}$$
$$= \lim_{h\to0}(4+h)$$

As $h \to 0$, $4 + h \to 4$.
So $f'(2) = 4$

b $$f'(-3) = \lim_{h\to0}\frac{f(-3+h)-f(-3)}{h}$$
$$= \lim_{h\to0}\frac{(-3+h)^2-(-3)^2}{h}$$
$$= \lim_{h\to0}\frac{9-6h+h^2-9}{h}$$
$$= \lim_{h\to0}\frac{-6h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(-6+h)}{h}$$
$$= \lim_{h\to0}(-6+h)$$

As $h \to 0$, $-6 + h \to -6$.
So $f'(-3) = -6$

c $$f'(0) = \lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$
$$= \lim_{h\to0}\frac{(0+h)^2-0^2}{h}$$
$$= \lim_{h\to0}\frac{h^2}{h}$$
$$= \lim_{h\to0} h$$

$f'(0) = 0$

d $$f'(50) = \lim_{h\to0}\frac{f(50+h)-f(50)}{h}$$
$$= \lim_{h\to0}\frac{(50+h)^2-50^2}{h}$$
$$= \lim_{h\to0}\frac{2500+100h+h^2-2500}{h}$$
$$= \lim_{h\to0}\frac{100h+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(100+h)}{h}$$
$$= \lim_{h\to0}(100+h)$$

As $h \to 0$, $100 + h \to 100$.
So $f'(50) = 100$

2 a $f(x) = x^2$

$$f'(x) = \lim_{h\to0}\frac{f(x+h)-f(x)}{h}$$
$$= \lim_{h\to0}\frac{(x+h)^2-x^2}{h}$$
$$= \lim_{h\to0}\frac{x^2+2xh+h^2-x^2}{h}$$
$$= \lim_{h\to0}\frac{2xh+h^2}{h}$$
$$= \lim_{h\to0}\frac{h(2x+h)}{h}$$
$$= \lim_{h\to0}(2x+h)$$

b As $h \to 0$, $2x + h \to 2x$.
So $f'(x) = 2x$

3 a $y = x^3$, therefore $f(x) = x^3$

$$g = \lim_{h\to0}\frac{f(-2+h)-f(-2)}{h}$$
$$= \lim_{h\to0}\frac{(-2+h)^3-(-2)^3}{h}$$
$$= \lim_{h\to0}\frac{-8+3(-2)^2h+3(-2)h^2+h^3+8}{h}$$
$$= \lim_{h\to0}\frac{12h-6h^2+h^3}{h}$$
$$= \lim_{h\to0}\frac{h(12-6h+h^2)}{h}$$
$$= \lim_{h\to0}(12-6h+h^2)$$

3 b As $h \to 0$, $12 - 6h + h^2 \to 12$.
So $g = 12$

4 a y-coordinate of point B
$= (-1 + h)^3 - 5(-1 + h)$
Gradient of AB

$$= \frac{y_2 - y_1}{x_2 - x_1}$$

$$= \frac{(-1+h)^3 - 5(-1+h) - 4}{(-1+h) - (-1)}$$

$$= \frac{-1 + 3h - 3h^2 + h^3 + 5 - 5h - 4}{h}$$

$$= \frac{h^3 - 3h^2 - 2h}{h}$$

$$= h^2 - 3h - 2$$

b At point A, as $h \to 0$, $h^2 - 3h - 2 \to -2$.
So gradient $= -2$

5 $f(x) = 6x$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{6(x+h) - 6x}{h}$$

$$= \lim_{h\to 0}\frac{6x + 6h - 6x}{h}$$

$$= \lim_{h\to 0}\frac{6h}{h}$$

$$= \lim_{h\to 0} 6$$

So $f'(x) = 6$

6 $f(x) = 4x^2$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{4(x+h)^2 - 4x^2}{h}$$

$$= \lim_{h\to 0}\frac{4x^2 + 8xh + 4h^2 - 4x^2}{h}$$

$$= \lim_{h\to 0}\frac{8xh + 4h^2}{h}$$

$$= \lim_{h\to 0}\frac{h(8x + 4h)}{h}$$

$$= \lim_{h\to 0}(8x + 4h)$$

As $h \to 0$, $8x + 4h \to 8x$.
So $f'(x) = 8x$

7 $f(z) = az^2$

$$f'(z) = \lim_{h\to 0}\frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0}\frac{a(z+h)^2 - az^2}{h}$$

$$= \lim_{h\to 0}\frac{az^2 + 2azh + ah^2 - az^2}{h}$$

$$= \lim_{h\to 0}\frac{2azh + ah^2}{h}$$

$$= \lim_{h\to 0}\frac{h(2az + ah)}{h}$$

$$= \lim_{h\to 0}(2az + ah)$$

As $h \to 0$, $2az + ah \to 2az$.
So $f'(z) = 2az$

Challenge

a $f(x) = \dfrac{1}{x}$

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0}\frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h\to 0}\frac{\frac{x - (x+h)}{x(x+h)}}{h}$$

$$= \lim_{h\to 0}\frac{-h}{xh(x+h)}$$

$$= \lim_{h\to 0}\frac{-1}{x(x+h)}$$

$$= \lim_{h\to 0}\frac{-1}{x^2 + hx}$$

b As $h \to 0$, $\dfrac{-1}{x^2 + hx} \to -\dfrac{1}{x^2}$.

So $f'(x) = -\dfrac{1}{x^2}$