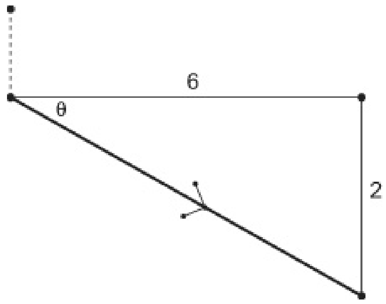
Vectors, 11 Mixed Exercise

1 a $\mathbf{R} = -3\mathbf{i} + 7\mathbf{j} + \mathbf{i} - \mathbf{j}$
$= -2\mathbf{i} + 6\mathbf{j}$
$|\mathbf{R}| = \sqrt{2^2 + 6^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$
The magnitude of **R** is $2\sqrt{10}$ N

b $\tan\theta = \frac{1}{3}$
$\theta = \tan^{-1}\frac{1}{3}$
$= 18°$ (nearest degree)

2 a $(\text{Path of } S) = (4\mathbf{i} - 6\mathbf{j}) - (-2\mathbf{i} - 4\mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$



$\tan\theta = \frac{1}{3} \Rightarrow \theta = 18.43\ldots°$
$\text{Bearing} = 90° + \theta = 108°$

b Expressing velocity, **v**, in km h^{-1}:
$\mathbf{v} = (6\mathbf{i} - 2\mathbf{j}) \times \frac{60}{40}$
$\mathbf{v} = 9\mathbf{i} - 3\mathbf{j}$
Then the speed is:
$\sqrt{9^2 + (-3)^2} = \sqrt{90} = 3\sqrt{10}$
Speed is 9.49 km h^{-1} (3 s.f.) to the left.

3 a Speed $= \sqrt{9^2 + 4^2}$
$= \sqrt{97}$
$= 9.85 \text{ m s}^{-1}$ (3 s.f.)

b Distance = speed × time
$= \sqrt{97} \times 6$
$= 59.1$ m

c This model becomes less accurate as t increases because it ignores friction and air resistance.

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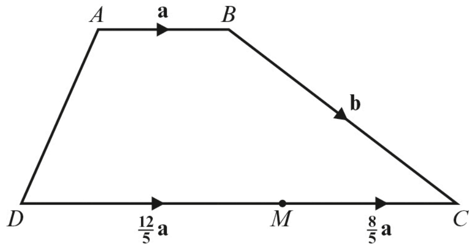

a $\overrightarrow{AM} = \overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CM}$
$= \mathbf{a} + \mathbf{b} - \frac{8}{5}\mathbf{a}$
$= \mathbf{b} - \frac{3}{5}\mathbf{a}$

b $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD}$
$= \mathbf{b} - 4\mathbf{a}$

c $\overrightarrow{MB} = \overrightarrow{MC} + \overrightarrow{CB}$
$= \frac{8}{5}\mathbf{a} - \mathbf{b}$

d $\overrightarrow{DA} = \overrightarrow{DC} + \overrightarrow{CB} + \overrightarrow{BA}$
$= 4\mathbf{a} - \mathbf{b} - \mathbf{a}$
$= 3\mathbf{a} - \mathbf{b}$

5 As the vectors are parallel
$5\mathbf{a} + k\mathbf{b} = \frac{5}{8}(8\mathbf{a} + 2\mathbf{b})$
$k = \frac{5}{8} \times 2$
$= 1.25$

6 a $\mathbf{a} + \mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} + \begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 12 \\ -1 \end{pmatrix}$

b $\mathbf{a} - 2\mathbf{b} + \mathbf{c} = \begin{pmatrix} 7 \\ 4 \end{pmatrix} - 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} + \begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} -18 \\ 5 \end{pmatrix}$

c $2\mathbf{a} + 2\mathbf{b} - 3\mathbf{c} = 2\begin{pmatrix} 7 \\ 4 \end{pmatrix} + 2\begin{pmatrix} 10 \\ -2 \end{pmatrix} - 3\begin{pmatrix} -5 \\ -3 \end{pmatrix}$
$= \begin{pmatrix} 49 \\ 13 \end{pmatrix}$

7 a $\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$
$= -(3\mathbf{i} + 5\mathbf{j}) + 6\mathbf{i} + 3\mathbf{j}$
$= 3\mathbf{i} - 2\mathbf{j}$

b $\tan x = \frac{5}{3}$
$x = \tan^{-1}\frac{5}{3}$
$= 59.036...$
$\tan y = \frac{1}{2}$
$y = \tan^{-1}\frac{1}{2}$
$= 26.565...$
$\angle BAC = 59.036... - 26.565...$
$= 32.5°$ (3 s.f.)

c Area $= \frac{1}{2}bc\sin A$
$b = \sqrt{6^2 + 3^2} = \sqrt{45}$
$c = \sqrt{3^2 + 5^2} = \sqrt{34}$
Area $= \frac{1}{2} \times \sqrt{45} \times \sqrt{34} \times \sin 32.5°$
$= 10.508...$
$= 10.5$ units^2 (3 s.f.)

8 a $4\mathbf{i} - 3\mathbf{j} + 2p\mathbf{i} - p\mathbf{j} = \lambda(2\mathbf{i} - 3\mathbf{j})$
$(4 + 2p)\mathbf{i} - (3 + p)\mathbf{j} = 2\lambda\mathbf{i} - 3\lambda\mathbf{j}$
Equating coefficients:
$4 + 2p = 2\lambda$ and $3 + p = 3\lambda$
Solving simultaneously:
Rearranging the $3 + p = 3\lambda$
$p = 3\lambda - 3$
Using substitution:
$4 + 2(3\lambda - 3) = 2\lambda$
$4 + 6\lambda - 6 = 2\lambda$
$4\lambda = 2$
$\lambda = \frac{1}{2}$
$p = -1.5$

b $\mathbf{a} + \mathbf{b} = 4\mathbf{i} - 3\mathbf{j} - 3\mathbf{i} + 1.5\mathbf{j}$
$= \mathbf{i} - 1.5\mathbf{j}$

9 a i a unit vector is $\frac{\mathbf{a}}{|\mathbf{a}|}$

$\mathbf{a} = \begin{pmatrix} 8 \\ 15 \end{pmatrix}$

$|\mathbf{a}| = \sqrt{8^2 + 15^2} = \sqrt{289} = 17$

9 a i $\frac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{17}\begin{pmatrix} 8 \\ 15 \end{pmatrix} = \frac{1}{17}(8\mathbf{i} + 15\mathbf{j})$

ii $\tan\theta = \frac{15}{8}$
$\theta = \tan^{-1}\frac{15}{8}$
$= 61.9°$ (3 s.f.) above

b i a unit vector is $\frac{\mathbf{b}}{|\mathbf{b}|}$

$\mathbf{b} = \begin{pmatrix} 24 \\ -7 \end{pmatrix}$

$|\mathbf{b}| = \sqrt{24^2 + 7^2}$
$= \sqrt{625}$
$= 25$

$\frac{\mathbf{b}}{|\mathbf{b}|} = \frac{1}{25}\begin{pmatrix} 25 \\ -7 \end{pmatrix}$

$= \frac{1}{25}(24\mathbf{i} - 7\mathbf{j})$

ii $\tan\theta = \frac{7}{24}$
$\theta = \tan^{-1}\frac{7}{24}$
$= 16.3°$ (3 s.f.) below

c i a unit vector is $\frac{\mathbf{c}}{|\mathbf{c}|}$

$\mathbf{c} = \begin{pmatrix} -9 \\ 40 \end{pmatrix}$

$|\mathbf{c}| = \sqrt{9^2 + 40^2}$
$= \sqrt{1681}$
$= 41$

$\frac{\mathbf{c}}{|\mathbf{c}|} = \frac{1}{41}\begin{pmatrix} -9 \\ 40 \end{pmatrix}$

$= \frac{1}{41}(-9\mathbf{i} + 40\mathbf{j})$

ii $\tan x = \frac{40}{9}$
$x = \tan^{-1}\frac{40}{9}$
$= 77.3°$ (3 s.f.)
$\theta = 180° - 77.3°$
$= 102.7°$ above

9 d i a unit vector is $\frac{\mathbf{d}}{|\mathbf{d}|}$

$$\mathbf{d} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$|\mathbf{d}| = \sqrt{3^2 + 2^2}$$

$$= \sqrt{13}$$

$$\frac{\mathbf{d}}{|\mathbf{d}|} = \frac{1}{\sqrt{13}}\begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

$$= \frac{1}{\sqrt{13}}(3\mathbf{i} - 2\mathbf{j})$$

ii $\tan\theta = \frac{2}{3}$

$$\theta = \tan^{-1}\frac{2}{3}$$

$= 33.7°$ (3 s.f.) below

10 $\cos 55° = \frac{p}{15}$

$$p = 15\cos 55°$$

$$p = 8.6$$

Using Pythagoras' theorem:

$$q = \sqrt{15^2 - 8.6^2}$$

$$= 12.3$$

$p = 8.6$ and $q = 12.3$

11 $|3\mathbf{i} - k\mathbf{j}| = \sqrt{3^2 + k^2}$

$$= \sqrt{9 + k^2}$$

$$= 3\sqrt{5}$$

$$\sqrt{9 + k^2} = \sqrt{45}$$

$$k^2 + 9 = 45$$

$$k^2 = 36$$

$$k = \pm 6$$

Q12 has been replaced in the summer 2020 impression of the book onwards. This is the answer to the updated question.

12 a $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$, $\overrightarrow{AP} = 3\overrightarrow{AB} = 3\mathbf{b} - 3\mathbf{a}$

$$\overrightarrow{MP} = \overrightarrow{MA} + \overrightarrow{AP}$$

$$= \frac{1}{2}\overrightarrow{OA} + \overrightarrow{AP}$$

$$= \frac{1}{2}\mathbf{a} + 3\mathbf{b} - 3\mathbf{a}$$

$$= 3\mathbf{b} - \frac{5}{2}\mathbf{a}$$

b $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$$= \frac{1}{2}\mathbf{a} + k\left(3\mathbf{b} - \frac{5}{2}\mathbf{a}\right)$$

$$= \left(\frac{1}{2} - \frac{5}{2}k\right)\mathbf{a} + 3k\mathbf{b}$$

c Since $\overrightarrow{ON}$ is parallel to **b**, component of **a** is 0:

$\frac{1}{2} - \frac{5}{2}k = 0 \Rightarrow k = \frac{1}{5}$ so $\overrightarrow{ON} = \frac{3}{5}\overrightarrow{OB}$,

so $ON : NB = 3 : 2$ as required.

This is the answer to Q12 from older impressions of the book.

12 a Using $\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$$\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$$

$$(1 - \lambda)\mathbf{a} + \lambda\mathbf{b} = \frac{3}{5}\mathbf{a} + \lambda\mathbf{b}$$

Equating coefficients

$$1 - \lambda = \frac{3}{5}$$

$$\lambda = \frac{2}{5}$$

$$\overrightarrow{ON} = \frac{3}{5}\mathbf{a} + \frac{2}{5}\mathbf{b}$$

b $\overrightarrow{MN} = \lambda\mathbf{b}$

$$= \frac{2}{5}\mathbf{b}$$

c $\overrightarrow{AN} = \frac{2}{5}(-\mathbf{a} + \mathbf{b})$

$$\overrightarrow{AB} = -\mathbf{a} + \mathbf{b}$$

Therefore, $AN : NB = 2 : 3$

13 a $\tan\theta = \frac{1}{3}$

$\theta = \tan^{-1}\frac{1}{3}$

$= 18.4°$ below

b $4\mathbf{i} - 5\mathbf{j} + p\mathbf{i} + q\mathbf{j} = \lambda(3\mathbf{i} - \mathbf{j})$
$(4 + p)\mathbf{i} + (q - 5)\mathbf{j} = 3\lambda\mathbf{i} - \lambda\mathbf{j}$
$4 + p = 3\lambda$ and $q - 5 = -\lambda$

Multiplying the second equation by 3:
$3q - 15 = -3\lambda$

Solving simultaneously:
$4 + p = -3q + 15$
$p + 3q = 11$

c When $p = 2$, $\lambda = 2$.
$\mathbf{R} = 2(3\mathbf{i} - \mathbf{j})$
$= 6\mathbf{i} - 2\mathbf{j}$
$|\mathbf{R}| = \sqrt{6^2 + 2^2}$
$= \sqrt{40}$
$= 2\sqrt{10}$ N

14 $\mathbf{v} - \mathbf{u} = (15\mathbf{i} - 3\mathbf{j}) - (3\mathbf{i} + 4\mathbf{j})$
$= 12\mathbf{i} - 7\mathbf{j}$
$|\mathbf{a}| = \frac{\sqrt{12^2 + 7^2}}{2}$
$= \frac{\sqrt{193}}{2}$

Challenge

$y = 5 - \frac{5}{3}x$

Using Pythagoras' theorem:
$x^2 + y^2 = \frac{17}{2}$

Solve the equations simultaneously.
Substitute $y = 5 - \frac{5}{3}x$ into $x^2 + y^2 = \frac{17}{2}$.

$$x^2 + \left(5 - \tfrac{5}{3}x\right)^2 = \tfrac{17}{2}$$
$$x^2 + 25 - \tfrac{50}{3}x + \tfrac{25}{9}x^2 - \tfrac{17}{2} = 0$$
$$18x^2 + 450 - 300x + 50x^2 - 153 = 0$$
$$68x^2 - 300x + 297 = 0$$

Using the quadratic formula:
$x =$

$x = \frac{300 \pm \sqrt{9216}}{136}$

$x = \frac{300 \pm 96}{136}$

$x = \frac{99}{34}$ or $x = \frac{51}{34}$

When $x = \frac{99}{34}$, $y = \frac{5}{34}$

When $x = \frac{51}{34}$, $y = \frac{5}{2}$

$\overrightarrow{OB} = \frac{99}{34}\mathbf{i} + \frac{5}{34}\mathbf{j}$ or $\overrightarrow{OB} = \frac{51}{34}\mathbf{i} + \frac{5}{2}\mathbf{j}$

