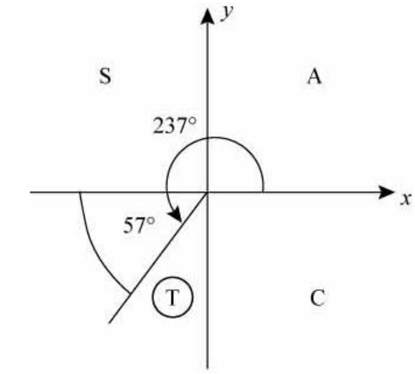
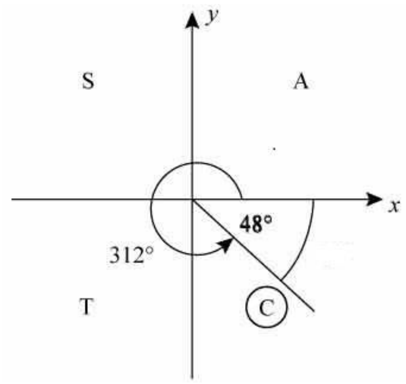
Trigonometric identities and equations, Mixed exercise 10

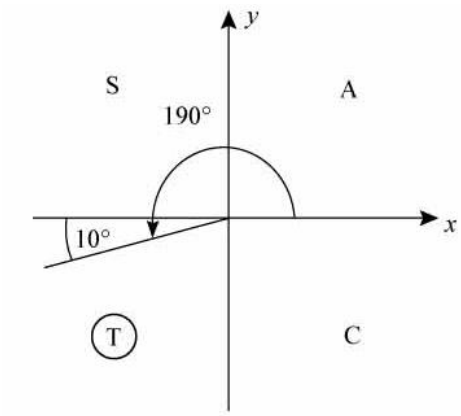
1 a 237° is in the third quadrant, so cos 237° is –ve.
The angle made with the horizontal is 57°.
So $\cos 237° = -\cos 57°$



b 312° is in the fourth quadrant so sin 312° is –ve.
The angle to the horizontal is 48°.
So $\sin 312° = -\sin 48°$



c 190° is in the third quadrant so tan 190° is +ve.
The angle to the horizontal is 10°.
So $\tan 190° = +\tan 10°$



2 a $\cos 270° = 0$

b $\sin 225° = \sin(180 + 45)°$
$= -\sin 45°$
$= -\frac{\sqrt{2}}{2}$

c $\cos 180° = -1$ $(\text{see graph of } y = \cos\theta)$

d $\tan 240° = \tan(180 + 60)°$
$= +\tan 60° \ (\text{third quadrant})$
So $\tan 240° = +\sqrt{3}$

e $\tan 135° = -\tan 45° \ (\text{second quadrant})$
So $\tan 135° = -1$

3 Using $\sin^2 A + \cos^2 A \equiv 1$

$$\sin^2 A + \left(-\sqrt{\frac{7}{11}}\right)^2 = 1$$

$$\sin^2 A = 1 - \frac{7}{11}$$

$$= \frac{4}{11}$$

$$\sin A = \pm\frac{2}{\sqrt{11}}$$

But A is in the second quadrant (obtuse),
so $\sin A$ is + ve.

So $\sin A = +\frac{2}{\sqrt{11}}$

Using $\tan A = \frac{\sin A}{\cos A}$

$$\tan A = \frac{\left(\frac{2}{\sqrt{11}}\right)}{-\sqrt{\frac{7}{11}}}$$

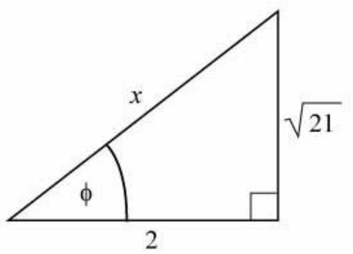
$$= -\frac{2}{\sqrt{11}} \times \frac{\sqrt{11}}{\sqrt{7}}$$

$$= -\frac{2}{\sqrt{7}}$$

$$= -\frac{2\sqrt{7}}{7}$$

(rationalising the denominator)

4 Draw a right-angled triangle with an angle of ϕ, where $\phi = +\frac{\sqrt{21}}{2}$.



Use Pythagoras' theorem to find the hypotenuse.

$$x^2 = 2^2 + \left(\sqrt{21}\right)^2$$
$$= 4 + 21$$
$$= 25$$

So $x = 5$

a $\sin\phi = \frac{\sqrt{21}}{5}$

As B is reflex and $\tan B$ is +ve, B is in the third quadrant.

So $\sin B = -\sin\phi$

$$= -\frac{\sqrt{21}}{5}$$

b From the diagram $\cos\phi = \frac{2}{5}$.

B is in the third quadrant. So

$\cos B = -\cos\phi$

$$= -\frac{2}{5}$$

5 a Factorise $\cos^4\theta - \sin^4\theta$.

(This is the difference of two squares.

$\cos^4\theta - \sin^4\theta$

$= \left(\cos^2\theta + \sin^2\theta\right)\left(\cos^2\theta - \sin^2\theta\right)$

$= (1)\left(\cos^2\theta - \sin^2\theta\right)$

$\left(\text{as} \cos^2\theta + \sin^2\theta \equiv 1\right)$

So $\cos^4\theta - \sin^4\theta = \cos^2\theta - \sin^2\theta$

b Factorise $\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$.

$\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta$

$= \sin^2 3\theta\left(1 - \cos^2 3\theta\right)$

5 b (use $\sin^2 3\theta + \cos^2 3\theta \equiv 1$)

$\sin^2 3\theta - \sin^2 3\theta\cos^2 3\theta = \sin^2 3\theta\left(\sin^2 3\theta\right)$

$= \sin^4 3\theta$

c $\cos^4\theta + 2\sin^2\theta\cos^2\theta + \sin^4\theta$

$= \left(\cos^2\theta + \sin^2\theta\right)^2$

$= 1$ (since $\sin^2\theta + \cos^2\theta \equiv 1$)

6 a $2(\sin x + 2\cos x) = \sin x + 5\cos x$

$\Rightarrow 2\sin x + 4\cos x = \sin x + 5\cos x$

$\Rightarrow 2\sin x - \sin x = 5\cos x - 4\cos x$

$\Rightarrow \sin x = \cos x$

(divide both sides by $\cos x$)

So $\tan x = 1$

b $\sin x\cos y + 3\cos x\sin y$

$= 2\sin x\sin y - 4\cos x\cos y$

$\Rightarrow \frac{\sin x\cos y}{\cos x\cos y} + \frac{3\cos x\sin y}{\cos x\cos y}$

$= \frac{2\sin x\sin y}{\cos x\cos y} - \frac{4\cos x\cos y}{\cos x\cos y}$

$\Rightarrow \tan x + 3\tan y = 2\tan x\tan y - 4$

$\Rightarrow 2\tan x\tan y - 3\tan y = 4 + \tan x$

$\Rightarrow \tan y(2\tan x - 3) = 4 + \tan x$

So $\tan y = \frac{4 + \tan x}{2\tan x - 3}$

7 a $\text{LHS} = \left(1 + 2\sin\theta + \sin^2\theta\right) + \cos^2\theta$

$= 1 + 2\sin\theta + 1$ since $\sin^2\theta + \cos^2\theta \equiv 1$

$= 2 + 2\sin\theta$

$= 2(1 + \sin\theta)$

$=$ RHS

b $\text{LHS} = \cos^4\theta + \sin^2\theta$

$= \left(\cos^2\theta\right)^2 + \sin^2\theta$

$= \left(1 - \sin^2\theta\right)^2 + \sin^2\theta$

$= 1 - 2\sin^2\theta + \sin^4\theta + \sin^2\theta$

$= \left(1 - \sin^2\theta\right) + \sin^4\theta$

$= \cos^2\theta + \sin^4\theta$

(using $\sin^2\theta + \cos^2\theta \equiv 1$)

$=$ RHS

8 a $\sin\theta = \frac{3}{2}$ has no solutions as $-1 \le \sin\theta \le 1$.

b $\sin\theta = -\cos\theta$
$\Rightarrow \tan\theta = -1$
Look at the graph of $y = \tan\theta$ in the interval $0 \le \theta \le 360°$. There are two solutions.

c The minimum value of $2\sin\theta$ is –2.
The minimum value of 3 $\cos\theta$ is -3.
Each minimum value is for a different θ.
So the minimum value of
$2\sin\theta + 3\cos\theta$ is always greater than -5.
There are no solutions of
$2\sin\theta + 3\cos\theta + 6 = 0$
as the LHS can never be zero.

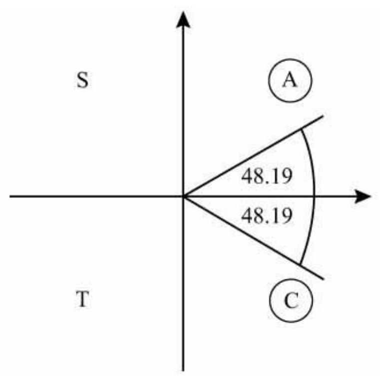
d Solving $\tan\theta + \frac{1}{\tan\theta} = 0$ is equivalent to solving $\tan^2\theta = -1$, which has no solutions.
So there are no solutions.

9 a $4xy - y^2 + 4x - y \equiv y(4x - y) + (4x - y)$
$= (4x - y)(y + 1)$

b Using **a** with $x = \sin\theta, y = \cos\theta$
$4\sin\theta\cos\theta - \cos^2\theta$
$+ 4\sin\theta - \cos\theta = 0$
So
$(4\sin\theta - \cos\theta)(\cos\theta + 1) = 0$
So $4\sin\theta - \cos\theta = 0$ or
$\cos\theta + 1 = 0$
$4\sin\theta - \cos\theta = 0$
$\Rightarrow \tan\theta = \frac{1}{4}$
The calculator solution is $\theta = 14.0°$.
$\tan\theta$ is +ve so θ is in the first and third quadrants.
So $\theta = 14.0°, 194°$
$\cos\theta + 1 = 0 \Rightarrow \cos\theta = -1$
So $\theta = +180°$ (from graph)
Solutions are $\theta = 14.0°, 180°, 194°$

10 a As $\sin(90 - \theta)° \equiv \cos\theta°$,
$\sin(90 - 3\theta)° \equiv \cos 3\theta°$
So $4\cos 3\theta° - \sin(90 - 3\theta)°$
$= 4\cos 3\theta° - \cos 3\theta$
$= 3\cos 3\theta°$

b Using **a**, $4\cos 3\theta° - \sin(90 - 3\theta)° = 2$
is equivalent to $3\cos 3\theta° = 2$
so $\cos 3\theta° = \frac{2}{3}$
Let $X = 3\theta$ and solve $\cos X° = \frac{2}{3}$
in the interval $0° \le X \le 1080°$.
The calculator solution is $X = 48.19°$
As $\cos X°$ is + ve, X is in the first and fourth quadrants.

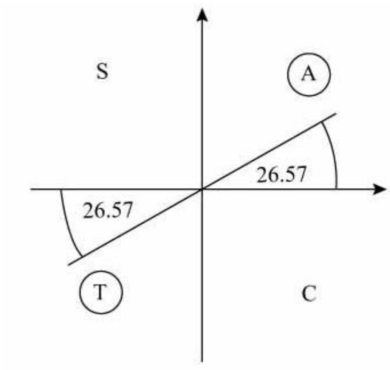


Read off all solutions in the interval $0° \le X \le 1080°$.
$X = 48.19°, 311.81°, 408.19°, 671.81°, 768.19°, 1031.81°$
So $\theta = \frac{X}{3} = 16.1°, 104, 136°, 224°, 256°,$
$344°$ (3 s.f.)

11 a $2\sin 2\theta = \cos 2\theta$
$\Rightarrow \frac{2\sin 2\theta}{\cos 2\theta} = 1$
$\Rightarrow 2\tan 2\theta = 1$ since $\tan 2\theta = \frac{\sin 2\theta}{\cos 2\theta}$
So $\tan 2\theta = 0.5$

b Solve $\tan 2\theta° = 0.5$ in the interval $0° \le \theta < 360°$ or $\tan X° = 0.5$ where $X = 2\theta,\ 0° \le X < 720°$.

11 b The calculator solution for $\tan^{-1} 0.5$ is $26.57°$.
As $\tan X$ is +ve, X is in the first and third quadrants.



Read off solutions for X in the interval $0° \le X < 720°$.

$X = 26.57°, 206.57°, 386.57°, 566.57°$

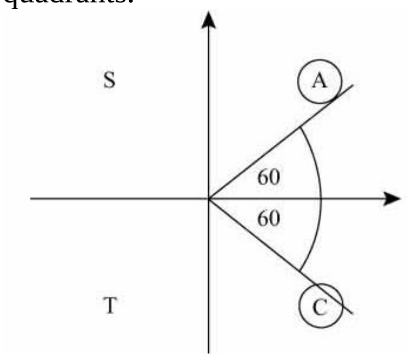
$= 2\theta$

So $\theta = \dfrac{X}{2}$

$= 13.3°, 103.3°, 193.3°, 283.3° \text{ (1 d.p.)}$

12 a $\cos(\theta + 75)° = 0.5$

Solve $\cos X° = 0.5$, where $X = \theta + 75$, $75° \le X < 435°$.
Your calculator solution for $X = 60°$.
As $\cos X$ is +ve, X is in the first and fourth quadrants.



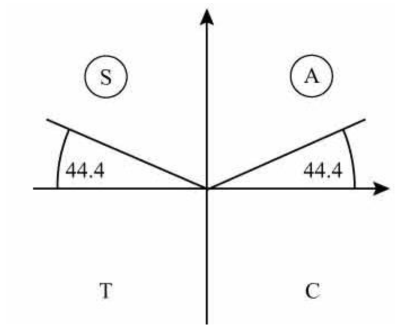
Read off all solutions in the interval $75° \le X < 435°$.

$X = 300°, 420°$

$\theta + 75° = 300°, 420°$

So $\theta = 225°, 345°$

12 b $\sin 2\theta° = 0.7$ in the interval $0° \le \theta < 360°$.
Solve $\sin X° = 0.7$, where $X = 2\theta$, $0° \le X < 720°$.
The calculator solution is $44.4°$.
As $\sin X$ is +ve, X is in the first and second quadrants.



Read off solutions in the interval $0° \le X < 720°$.

$X = 44.4°, 135.6°, 404.4°, 495.6°$

$= 2\theta$

So $\theta = \dfrac{X}{2}$

$= 22.2°, 67.8°, 202.2°, 247.8° \text{ (1 d.p.)}$

13 Multiply both sides of the equation by $(1 - \cos 2x)$, provided $\cos 2x \ne 1$.
Note: In the interval given, $\cos 2x$ is never equal to 1.

So $\cos 2x + 0.5 = 2 - 2\cos 2x$

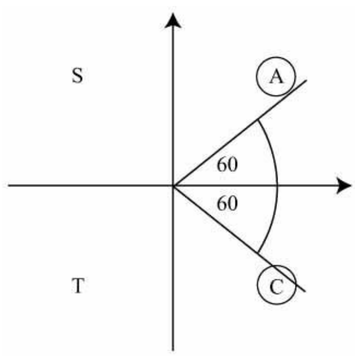
$\Rightarrow 3\cos 2x = \dfrac{3}{2}$

So $\cos 2x = \dfrac{1}{2}$

Solve $\cos X = \dfrac{1}{2}$ where $X = 2x$, $0° < X < 540°$.

The calculator solution is $60°$.

As $\cos X$ is + ve, X is in the first and fourth quadrants.



13 Read off solutions for X in the interval $0° < X < 540°$.

$X = 60°, 300°, 420°$

So $x = \frac{X}{2}$

$= 30°, 150°, 210°$

14 Using $\sin^2\theta + \cos^2\theta \equiv 1$

$2\cos^2\theta - \cos\theta - 1 = 1 - \cos^2\theta$

$\Rightarrow 3\cos^2\theta - \cos\theta - 2 = 0$

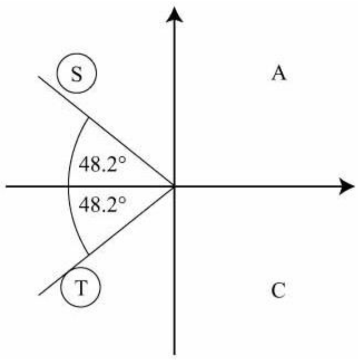
$\Rightarrow (3\cos\theta + 2)(\cos\theta - 1) = 0$

So $3\cos\theta + 2 = 0$ or $\cos\theta - 1 = 0$

For $3\cos\theta + 2 = 0$, $\cos\theta = -\frac{2}{3}$

The calculator solution is 131.8°.

As $\cos\theta$ is –ve, θ is in the second a and third quadrants.



$\theta = 131.8°, 228.2°$

For $\cos\theta = 1$, $\theta = 0°$

(See graph and check the given interval.)

So solutions are

$\theta = 0°, 131.8°, 228.2°, 360°$

15 a The student found additional solutions after dividing by three rather than before. The students has not applied the full interval for the solutions.

b Let $X = 3x$

$\sin X = \frac{1}{2}$

As $X = 3x$, then as $-360° \le x \le 360°$

So $3 \times -360° \le X \le 3 \times 360°$

So the interval for X is

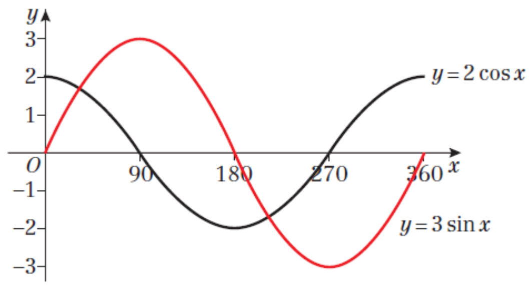
$-1080° \le X \le 1080°$

15 b $X = 30°, 150°, 390°, 510°, 750°, 870°, -210°, -330°, -570°, -690°, -930°, -1050°$

i.e. $3x = -1050°, -930°, -690°, -570°, -330°, -210°, 30°, 150°, 390°, 510°, 750°, 870°$

So $x = -350°, -310°, -230°, -190°, -110°, -70°, 10°, 50°, 130°, 170°, 250°, 290°$

16 a



b The graphs intersect at two places so there are two solutions to the equation in the given range.

c $3\sin x = 2\cos x$

$\frac{\sin x}{\cos x} = \frac{2}{3}$

$\tan x = \frac{2}{3}$

$x = 33.7°, 213.7°$

17 a Using the cosine rule

$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$

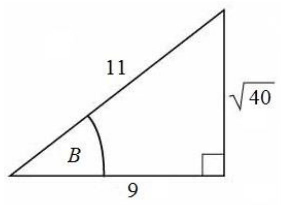
$\cos B = \frac{6^2 + 11^2 - 7^2}{2 \times 6 \times 11}$

$\cos B = \frac{36 + 121 - 49}{132}$

$\cos B = \frac{9}{11}$

b Using Pythagoras' theorem

$\sqrt{11^2 - 9^2} = \sqrt{40}$



$\sin B = \frac{\sqrt{40}}{11} = \frac{2\sqrt{10}}{11}$

18 a Using the sine rule

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$

$$\frac{\sin Q}{6} = \frac{\sin 45°}{5}$$

$$\sin Q = \frac{6\left(\sqrt{2}/2\right)}{5}$$

$$\sin Q = \frac{3\sqrt{2}}{5}$$

b Using Pythagoras' theorem or identity

$$\cos^2 x + \sin^2 x = 1$$

$\cos Q = \frac{\sqrt{7}}{5}$ for the acute angle

As Q is obtuse, it is in the second quadrant where cos Q is negative.

So $\cos Q = -\frac{\sqrt{7}}{5}$

19 a $3\sin^2 x - \cos^2 x = 2$ can be written as
$3\sin^2 x - (1 - \sin^2 x) = 2$
which reduces to
$4\sin^2 x = 3$

b $4\sin^2 x = 3$

$\sin^2 x = \frac{3}{4}$

$\sin x = \pm\frac{\sqrt{3}}{2}$

$x = 60°, 120°, -60°, -120°$
So the solutions are
$x = -120.0°, -60.0°, 60.0°, 120.0°$

20 $3\cos^2 x + 1 = 4\sin x$ can be written as
$3(1 - \sin^2 x) + 1 = 4\sin x$
which can be reduced to
$3\sin^2 x + 4\sin x - 4 = 0$
$(3\sin x - 2)(\sin x + 2) = 0$

$\sin x = \frac{2}{3}$ or $\sin x = -2$

$\sin x = -2$ has no solutions.
So $x = 41.8°, 138.2°, -221.8°, -318.2°$
So the solutions are
$x = -318.2°, -221.8°, 41.8°, 138.2°$

21 a Solving equation where $X = 2x + k$

$$3 + \sqrt{3} = 3 + 2\sin(X)$$

$$\sqrt{3} = 2\sin(X)$$

$$X = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

$$X = 60° \text{ or } 180° - 60° = 120°$$

$$k = 60° - 30° \text{ or } 120° - 30°$$

$$k = 30° \text{ or } 90°$$

b Solving equation where $X = 2x + 30$ and $30 \le X \le 750$

$$1 = 3 + 2\sin(X)$$

$$-2 = 2\sin(X)$$

$$X = \sin^{-1}(-1)$$

$$X = 270° \text{ or } 270° + 360° = 630°$$

$$x = \frac{X - 30°}{2}$$

$$x = 120° \text{ or } 300°$$

Challenge

$\tan^4 x - 3\tan^2 x + 2 = 0$
$(\tan^2 x - 2)(\tan^2 x - 1) = 0$
$\tan^2 x = 2$ or $\tan^2 x = 1$
$\tan x = \pm 1$ or $\pm\sqrt{2}$
$x = 45°, 225°, -45°, 135°, 315°, 54.7°, 234.7°, -54.7°, 125.3°, 305.3°$
So the solutions are
$x = 45°, 54.7°, 125.3°, 135°, 225°, 234.7°, 305.3°, 315°$