## Trigonometric identities and equations 10F

**1 a** $4\cos^2\theta = 1 \Rightarrow \cos^2\theta = \frac{1}{4}$

So $\cos\theta = \pm\frac{1}{2}$

Solutions are $60°, 120°, 240°, 300°$.

**b** $2\sin^2\theta - 1 = 0 \Rightarrow \sin^2\theta = \frac{1}{2}$

So $\sin\theta = \pm\frac{1}{\sqrt{2}}$

Solutions are in all four quadrants, at 45° to the horizontal.

So $\theta = 45°, 135°, 225°, 315°$

**c** Factorising, $\sin\theta(3\sin\theta + 1) = 0$

So $\sin\theta = 0$ or $\sin\theta = -\frac{1}{3}$

Solutions of $\sin\theta = 0$ are

$\theta = 0°\ 180°, 360°$ (from graph)

Solutions of $\sin\theta = -\frac{1}{3}$ are

$\theta = 199°,\ 341°$ (3 s.f. )

These are in the third and fourth quadrants.

**d** $\tan^2\theta - 2\tan\theta - 10 = 0$

So $\tan\theta = \frac{2 \pm \sqrt{4+40}}{2}$

$= \frac{2 \pm \sqrt{44}}{2}$

$(= -2.3166\ldots$ or $4.3166\ldots)$

Solutions of $\tan\theta = \frac{2-\sqrt{44}}{2}$ are in the second and fourth quadrants.

So $\theta = 113.35°, 293.3°$

Solutions of $\tan\theta = \frac{2+\sqrt{44}}{2}$ are in the first and third quadrants.

So $\theta = 76.95\ldots°, 256.95\ldots°$

Solution set: $77.0°, 113°, 257°, 293°$

**e** Factorising LHS of

$2\cos^2\theta - 5\cos\theta + 2 = 0$

$(2\cos\theta - 1)(\cos\theta - 2) = 0$

So $2\cos\theta - 1 = 0$ or $\cos\theta - 2 = 0$

As $\cos\theta \le 1, \cos\theta = 2$ has no solutions.

Solutions of $\cos\theta = \frac{1}{2}$ are $\theta = 60°,\ 300°$

**f** $\sin^2\theta - 2\sin\theta - 1 = 0$

So $\sin\theta = \frac{2 \pm \sqrt{8}}{2}$

Solve $\sin\theta = \frac{2-\sqrt{8}}{2}$ as $\frac{2+\sqrt{8}}{2} > 1$

$\theta = 204°, 336°$

The solutions are in the third and fourth quadrants as $\frac{2-\sqrt{8}}{2} < 0$.

**g** $\tan^2 2\theta = 3 \Rightarrow \tan 2\theta = \pm\sqrt{3}$

Solve $\tan X = +\sqrt{3}$ and $\tan X = -\sqrt{3}$, where $X = 2\theta$

The interval for $X$ is $0 \le X \le 720°$.

For $\tan X = \sqrt{3}$,

$X = 60°, 240°, 420°, 600°$

So $\theta = \frac{X}{2} = 30°, 120°, 210°, 300°$

For $\tan X = -\sqrt{3}$,

$X = 120°, 300°, 480°, 660°$

So $\theta = 60°, 150°, 240°, 330°$

Solution set:

$\theta = 30°, 60°, 120°, 150°, 210°, 240°,$
$300°, 330°$

**2 a** Solve $\sin^2 X = 1$ where $X = 2\theta$

The interval for $X$ is $-360° \le X \le 360°$.

$\sin X = +1$ gives $X = -270°, 90°$

$\sin X = -1$ gives $X = -90°, +270°$

$X = -270°, -90°, +90°, +270°$

So $\theta = \frac{X}{2}$

$= \pm 45°, \pm 135°$

**2 b** $\tan^2\theta = 2\tan\theta$

$\Rightarrow \tan^2\theta - 2\tan\theta = 0$

$\Rightarrow \tan\theta(\tan\theta - 2) = 0$

So $\tan\theta = 0$ or $\tan\theta = 2$

(first and third quadrants)

Solutions are $(-180°, 0°, 180°)$

and $(-116.6°, 63.4°)$.

Solution set:

$-180°, -117°, 0°, 63.4°, 180°$

**c** $\cos^2\theta - 2\cos\theta = 1$

$\Rightarrow \cos^2\theta - 2\cos\theta - 1 = 0$

So $\cos\theta = \frac{2 \pm \sqrt{8}}{2}$

$\Rightarrow \cos\theta = \frac{2-\sqrt{8}}{2}\left(\text{as } \frac{2+\sqrt{8}}{2} > 1\right)$

Solutions are $\pm 114°$

(second and third quadrants).

**d** $4\sin\theta = \tan\theta$

So $4\sin\theta = \frac{\sin\theta}{\cos\theta}$

$\Rightarrow 4\sin\theta\cos\theta = \sin\theta$

$\Rightarrow 4\sin\theta\cos\theta - \sin\theta = 0$

$\Rightarrow \sin\theta(4\cos\theta - 1) = 0$

So $\sin\theta = 0$ or $\cos\theta = \frac{1}{4}$

Solutions of $\cos\theta = \frac{1}{4}$ are

$\cos^{-1}\left(\frac{1}{4}\right)$ and $360° - \cos^{-1}\left(\frac{1}{4}\right)$

Solution set:

$0°, \pm 75.5°, \pm 180°$

**3 a** $4\sin^2\theta - 4\cos\theta = 3 - 2\cos\theta$

$\Rightarrow 4(1-\cos^2\theta) - 4\cos\theta = 3 - 2\cos\theta$

$\Rightarrow 4\cos^2\theta + 2\cos\theta - 1 = 0$

So $\cos\theta = \frac{-2 \pm \sqrt{20}}{8} = \left(\frac{-1 \pm \sqrt{5}}{4}\right)$

**3 a** Solutions of $\cos\theta = \frac{-2+\sqrt{20}}{8}$ are

Solutions of $\cos\theta = \frac{-2-\sqrt{20}}{8}$ are

$144°, -144°$ (second and third quadrants).

Solution set: $72°, 144°$

**b** $2\sin^2\theta = 3(1-\cos\theta)$

$\Rightarrow 2(1-\cos^2\theta) = 3(1-\cos\theta)$

$\Rightarrow 2(1-\cos\theta)(1+\cos\theta) = 3(1-\cos\theta)$

(or write as $a\cos^2\theta + b\cos\theta + c \equiv 0$)

$\Rightarrow (1-\cos\theta)(2(1+\cos\theta) - 3) = 0$

$\Rightarrow (1-\cos\theta)(2\cos\theta - 1) = 0$

So $\cos\theta = 1$ or $\cos\theta = \frac{1}{2}$

Solution of $\cos\theta = 1$ is $0°$

Solution of $\cos\theta = \frac{1}{2}$ are $-60°$, $60°$

Solution set: 0°, 60’

**c** $4\cos^2\theta - 5\sin\theta - 5 = 0$

$\Rightarrow 4(1-\sin^2\theta) - 5\sin\theta - 5 = 0$

$\Rightarrow 4\sin^2\theta + 5\sin\theta + 1 = 0$

$\Rightarrow (4\sin\theta + 1)(\sin\theta + 1) = 0$

So $\sin\theta = -1$ or $\sin\theta = -\frac{1}{4}$

Solution of $\sin\theta = -1$ is $\theta = 270°$.

Solution of $\sin\theta = -\frac{1}{4}$ are

$\theta = 194°, 346°$ (3 s.f.)

(second and fourth quadrants).

Solution set: the empty set

None of the solutions are in the required range.

**4 a** $5\sin^2\theta = 4\cos^2\theta$

$\Rightarrow \tan^2\theta = \frac{4}{5}$ as $\tan\theta = \frac{\sin\theta}{\cos\theta}$

So $\tan\theta = \pm\sqrt{\frac{4}{5}}$

**4 a** There are solutions from each of the quadrants $\left(\text{angle to horizontal is } 41.8°\right)$.

$\theta = \pm 138°, \pm 41.8°$

**b** $\tan\theta = \cos\theta$

$\Rightarrow \frac{\sin\theta}{\cos\theta} = \cos\theta$

$\Rightarrow \sin\theta = \cos^2\theta$

$\Rightarrow \sin\theta = 1 - \sin^2\theta$

$\Rightarrow \sin^2\theta + \sin\theta - 1 = 0$

So $\sin\theta = \frac{-1 \pm \sqrt{5}}{2}$

There are only solutions from

$\sin\theta = \frac{-1+\sqrt{5}}{2} \quad \left(\text{as } \frac{-1-\sqrt{5}}{2} < -1\right)$

Solutions are $\theta = 38.2°, 142°$

$\left(\text{first and second quadrants}\right)$.

**5** $8\sin^2 x + 6\cos x - 9 = 0$ can be written as
$8(1 - \cos^2 x) + 6\cos x - 9 = 0$
which reduces to
$8\cos^2 x - 6\cos x + 1 = 0$
So $(4\cos x - 1)(2\cos x - 1) = 0$
$\cos x = \frac{1}{4}$ or $\cos x = \frac{1}{2}$
So $x = 75.5°, 284.5°, 60°, 300°$

The solutions are
$x = 60°, 75.5°, 284.5°, 300°$

**6** $\sin^2 x + 1 = \frac{7}{2}\cos^2 x$ can be written as
$\sin^2 x + 1 = \frac{7}{2}\left(1 - \sin^2 x\right)$
$2\sin^2 x + 2 = 7 - 7\sin^2 x$ which reduces to
$9\sin^2 x - 5 = 0$
$\sin^2 x = \frac{5}{9}$
$\sin x = \pm\frac{\sqrt{5}}{3}$
So $x = 48.2°, 131.8°, -48.2°, 228.2, 311.8°$.
The solutions are
$x = 48.2°, 131.8°, 228.2, 311.8°$

**7** $2\cos^2 x + \cos x - 6 = 0$
$(2\cos x - 3)(\cos x + 2) = 0$
$\cos x = \frac{3}{2}$ or $\cos x = -2$

There are no solutions to $\cos x = \frac{3}{2}$ or $\cos x = -2$, so the equation has no solutions.

**8 a** $\cos^2 x = 2 - \sin x$ can be written as
$(1 - \sin^2 x) = 2 - \sin x$
$\sin^2 x - \sin x + 1 = 0$

**b** $\sin^2 x - \sin x + 1 = 0$
Using the discriminant
$b^2 - 4ac = (-1)^2 - 4 \times 1 \times 1$
$= -3$

As $b^2 - 4ac < 0$, therefore there are no real roots.

**9 a** $\tan^2 x - 2\tan x - 4 = 0$

$\tan x = \frac{2 \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$

$= \frac{2 \pm \sqrt{20}}{2}$

$= \frac{2 \pm 2\sqrt{5}}{2}$

$= 1 \pm \sqrt{5}$

$p = 1, q = 5$

**b** $\tan x = 1 \pm \sqrt{5}$
$x = 72.8°, 252.8°, 432.8°, -51.0°, 129.0°, 309.0°, 489.0°$

So the solutions are
$x = 72.8°, 129.0°, 252.8°, 309.0°, 432.8°, 489.0°$

**Challenge**

**a** Let $X = 3\theta$
So $\cos^2 X - \cos X - 2 = 0$
$(\cos X + 1)(\cos X - 2) = 0$
$\cos X = -1$ or $\cos X = 2$
$\cos X = 2$ has no solutions so $\cos X = -1$
As $X = 3\theta$, then as $-180° \leq \theta \leq 180°$
So $3 \times -180° \leq X \leq 3 \times 180°$
So the interval for $X$ is $-540° \leq X \leq 540°$.
$X = -540°, -180°, 180°, 540°$
I.e. $3\theta = -540°, -180°, 180°, 540°$
So $\theta = -180°, -60°, 60°, 180°$

**b** Let $X = \theta - 45°$
So $\tan^2 X = 1$
$\tan X = \pm 1$
As $X = \theta - 45°$, then as $0 \leq \theta \leq 360°$
So $0 - 45° \leq X \leq 360° - 45°$
So the interval for $X$ is $-45° \leq X \leq 315°$.
$X = -45°, 135°, 315°, 45°, 225°$
I.e. $\theta - 45° = -45°, 45°, 135°, 225°, 315°$
So $\theta = 0°, 90°, 180°, 270°, 360°$