## Trigonometric identities and equations 10C

**1 a** As $\sin^2 \frac{1}{2}\theta + \cos^2 \frac{1}{2}\theta \equiv 1$

So $1 - \cos^2 \frac{1}{2}\theta = \sin^2 \frac{1}{2}\theta$

**b** As $\sin^2 3\theta + \cos^2 3\theta \equiv 1$

So:

$$5\sin^2 3\theta + 5\cos^2 3\theta = 5\left(\sin^2 3\theta + \cos^2 3\theta\right) = 5$$

**c** As $\sin^2 A + \cos^2 A \equiv 1$

So $\sin^2 A - 1 \equiv -\cos^2 A$

**d** $$\frac{\sin\theta}{\tan\theta} = \frac{\sin\theta}{\sin\theta/\cos\theta} = \sin\theta \times \frac{\cos\theta}{\sin\theta} = \cos\theta$$

**e** $$\frac{\sqrt{1-\cos^2 x}}{\cos x} = \frac{\sqrt{\sin^2 x}}{\cos x} = \frac{\sin x}{\cos x} = \tan x$$

**f** $$\frac{\sqrt{1-\cos^2 3A}}{\sqrt{1-\sin^2 3A}} = \frac{\sqrt{\sin^2 3A}}{\sqrt{\cos^2 3A}} = \frac{\sin 3A}{\cos 3A} = \tan 3A$$

**g** $(1+\sin x)^2 + (1-\sin x)^2 + 2\cos^2 x$

$= 1 + 2\sin x + \sin^2 x + 1 - 2\sin x + \sin^2 x + 2\cos^2 x$

$= 2 + 2\sin^2 x + 2\cos^2 x$

$= 2 + 2\left(\sin^2 x + \cos^2 x\right)$

$= 2 + 2$

$= 4$

**h** $\sin^4\theta + \sin^2\theta\cos^2\theta$

$= \sin^2\theta\left(\sin^2\theta + \cos^2\theta\right)$

$= \sin^2\theta$

**1 i** $\sin^4\theta + 2\sin^2\theta\cos^2 + \cos^4\theta$

$= \left(\sin^2\theta + \cos^2\theta\right)^2$

$= 1^2$

$= 1$

**2** Given $2\sin\theta = 3\cos\theta$

So $\frac{\sin\theta}{\cos\theta} = \frac{3}{2}$

(divide both side by $2\cos\theta$)

So $\tan\theta = \frac{3}{2}$

**3** As $\sin x \cos y = 3\cos x \sin y$

So $\frac{\sin x \cos y}{\cos x \cos y} = 3\frac{\cos x \sin y}{\cos x \cos y}$

So $\tan x = 3\tan y$

**4 a** As $\sin^2\theta + \cos^2\theta \equiv 1$

So $\cos^2\theta \equiv 1 - \sin^2\theta$

**b** $$\tan^2\theta \equiv \frac{\sin^2\theta}{\cos^2\theta} \equiv \frac{\sin^2\theta}{1-\sin^2\theta}$$

**c** $$\cos\theta\tan\theta = \cos\theta \times \frac{\sin\theta}{\cos\theta} = \sin\theta$$

**d** $$\frac{\cos\theta}{\tan\theta} = \frac{\cos\theta}{\sin\theta/\cos\theta} = \cos\theta \times \frac{\cos\theta}{\sin\theta} = \frac{\cos^2\theta}{\sin\theta}$$

So $\frac{\cos\theta}{\tan\theta} = \frac{1-\sin^2\theta}{\sin\theta}$ or $\frac{1}{\sin\theta} - \sin\theta$

**e** $(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)$

$= \cos^2\theta - \sin^2\theta$

$= \left(1 - \sin^2\theta\right) - \sin^2\theta$

$= 1 - 2\sin^2\theta$

**5 a** $\text{LHS} = (\sin\theta + \cos\theta)^2$

$= \sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta$

$= (\sin^2\theta + \cos^2\theta) + 2\sin\theta\cos\theta$

$= 1 + 2\sin\theta\cos\theta$

$= \text{RHS}$

**b** $\text{LHS} = \dfrac{1}{\cos\theta} - \cos\theta$

$= \dfrac{1-\cos^2\theta}{\cos\theta}$

$= \dfrac{\sin^2\theta}{\cos\theta}$

$= \sin\theta \times \dfrac{\sin\theta}{\cos\theta}$

$= \sin\theta\tan\theta$

$= \text{RHS}$

**c** $\text{LHS} = \tan x + \dfrac{1}{\tan x}$

$= \dfrac{\sin x}{\cos x} + \dfrac{\cos x}{\sin x}$

$= \dfrac{\sin^2 x + \cos^2 x}{\sin x\cos x}$

$= \dfrac{1}{\sin x\cos x}$

$= \text{RHS}$

**d** $\text{LHS} = \cos^2 A - \sin^2 A$

$\equiv \cos^2 A - (1-\cos^2 A)$

$\equiv \cos^2 A - 1 + \cos^2 A$

$\equiv 2\cos^2 A - 1$ ✓

$\equiv 2(1-\sin^2 A) - 1$

$\equiv 2 - 2\sin^2 A - 1$

$\equiv 1 - 2\sin^2 A$ ✓

**e** $\text{LHS} = (2\sin\theta - \cos\theta)^2 + (\sin\theta + 2\cos\theta)^2$

$\equiv 4\sin^2\theta - 4\sin\theta\cos\theta + \cos^2\theta$

$+ \sin^2\theta + 4\sin\theta\cos\theta + 4\cos^2\theta$

$\equiv 5\sin^2\theta + 5\cos^2\theta$

$\equiv 5(\sin^2\theta + \cos^2\theta)$

$\equiv 5$

$\equiv \text{RHS}$

**5 f** $\text{LHS} = 2 - (\sin\theta - \cos\theta)^2$

$= 2 - (\sin^2\theta - 2\sin\theta\cos\theta + \cos^2\theta)$

$= 2 - (1 - 2\sin\theta\cos\theta)$

$= 1 + 2\sin\theta\cos\theta$

$= \sin^2\theta + \cos^2\theta + 2\sin\theta\cos\theta$

$= (\sin\theta + \cos\theta)^2$

$= \text{RHS}$

**g** $\text{LHS} = \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$

$= \sin^2 x(1-\sin^2 y)$

$-(1-\sin^2 x)\sin^2 y$

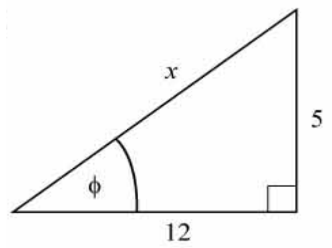
$= \sin^2 x - \sin^2 x\sin^2 y$

$-\sin^2 y + \sin^2 x\sin^2 y$

$= \sin^2 x - \sin^2 y$

$= \text{RHS}$

**6 a**



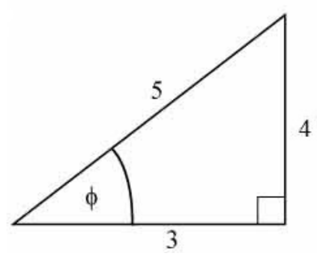

Using Pythagoras' theorem:

$x^2 = 12^2 + 5^2 = 169$

$x = 13$

So $\sin\theta = \dfrac{5}{13}$ and $\cos\theta = \dfrac{12}{13}$

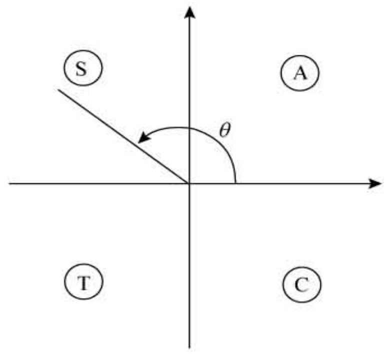
**b**




Using Pythagoras' theorem, $x = 4$

So $\sin\phi = \frac{4}{5}$ and $\tan\phi = -\frac{4}{3}$

**6 b**



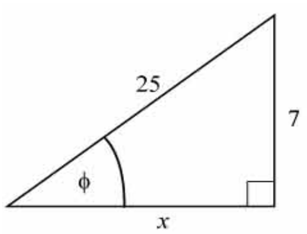

As $\theta$ is obtuse:

$\sin\theta = \sin\phi = \frac{4}{5}$

and

$\tan\theta = -\tan\phi = -\frac{4}{3}$

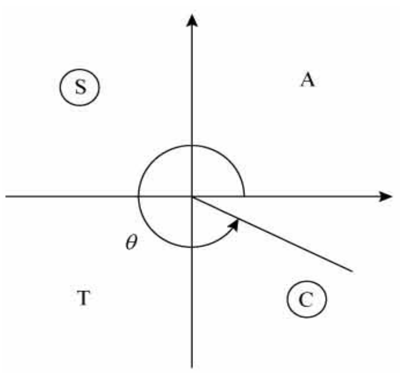
**c**




Using Pythagoras' theorem

$$x^2 + 7^2 = 25^2$$
$$x^2 = 25^2 - 7^2$$
$$= 576$$
$$x = 24$$

So $\cos\phi = \frac{24}{25}$ and $\tan\phi = \frac{7}{24}$



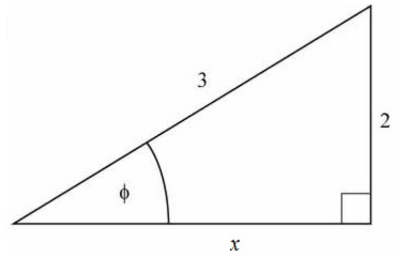

As $\theta$ is in the fourth quadrant

$$\cos\theta = +\cos\phi$$
$$= \frac{24}{25}$$

and $\tan\theta = -\tan\phi$

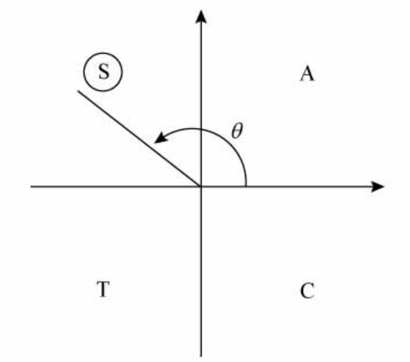
$$= -\frac{7}{24}$$

**7** Consider the angle $\phi$ where $\sin\phi = \frac{2}{3}$.




Using Pythagoras' theorem, $x = \sqrt{5}$

**a** So $\cos\phi = \dfrac{\sqrt{5}}{3}$




As $\theta$ is obtuse, $\cos\theta = -\cos\phi = -\dfrac{\sqrt{5}}{3}$

**b** From the triangle

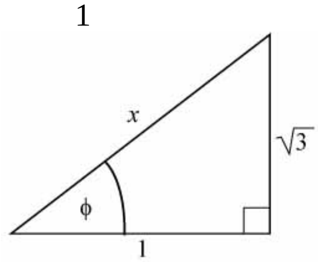
$$\tan\phi = \frac{2}{\sqrt{5}}$$
$$= \frac{2\sqrt{5}}{5}$$

Using the quadrant diagram

$$\tan\theta = -\tan\phi$$
$$= -\frac{2\sqrt{5}}{5}$$

**8** Draw a right-angled triangle with

$$\tan\phi = +\sqrt{3}$$
$$= \frac{\sqrt{3}}{1}$$



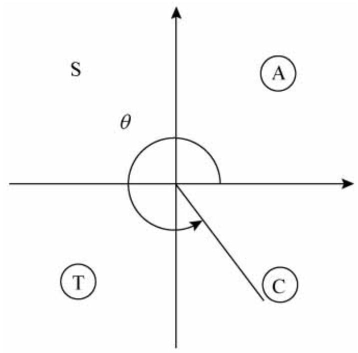

Using Pythagoras' theorem

$$x^2 = \left(\sqrt{3}\right)^2 + 1^2 = 4$$

So $x = 2$

**8 a** $\sin\phi = \frac{\sqrt{3}}{2}$




As $\theta$ is reflex and $\tan\phi$ is –ve, $\phi$ is in the fourth quadrant.

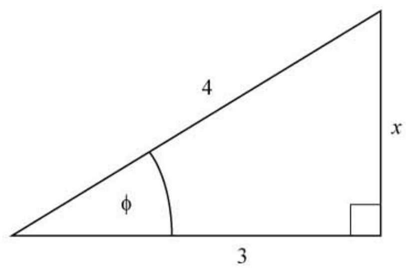
So $\sin\theta = -\sin\phi$

$= \frac{-\sqrt{3}}{2}$

**b** $\cos\phi = \frac{1}{2}$

As $\cos\theta = \cos\phi$, $\cos\theta = \frac{1}{2}$

**9** Draw a right-angled triangle with $\cos\phi = \frac{3}{4}$.




Using Pythagoras' theorem

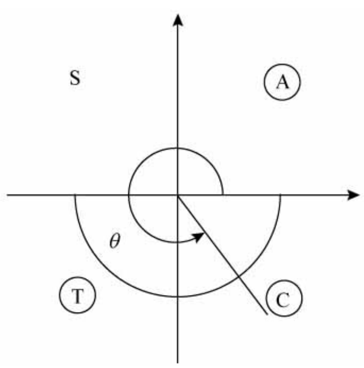
$x^2 + 3^2 = 4^2$

$x^2 = 4^2 - 3^2$

$= 7$

$x = \sqrt{7}$

So $\sin\phi = \frac{\sqrt{7}}{4}$ and $\tan\phi = \frac{\sqrt{7}}{3}$




As $\theta$ is reflex and $\cos\theta$ is +ve, $\theta$ is in the fourth quadrant.

**9 a** $\sin\theta = -\sin\phi$

$= -\frac{\sqrt{7}}{4}$

**b** $\tan\theta = -\tan\phi$

$= -\frac{\sqrt{7}}{3}$

**10 a** As $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + y^2 = 1$

**b** $\sin\theta = x$ and $\cos\theta = \frac{y}{2}$

So, using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + \left(\frac{y}{2}\right)^2 = 1$

or $x^2 + \frac{y^2}{4} = 1$

or $4x^2 + y^2 = 4$

**c** As $\sin\theta = x$

$\sin^2\theta = x^2$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + y = 1$

**d** As $\tan\theta = \frac{\sin\theta}{\cos\theta}$

$\cos\theta = \frac{\sin\theta}{\tan\theta}$

So $\cos\theta = \frac{x}{y}$

Using $\sin^2\theta + \cos^2\theta \equiv 1$

$x^2 + \frac{x^2}{y^2} = 1$ or $x^2y^2 + x^2 = y^2$

**e** $\sin\theta + \cos\theta = x$

$-\sin\theta + \cos\theta = y$

Adding the two equations:

$2\cos\theta = x + y$

So $\cos\theta = \frac{x+y}{2}$

Subtracting the two equations:

$2\sin\theta = x - y$

So $\sin\theta = \frac{x-y}{2}$

**10 e** Using $\sin^2\theta + \cos^2\theta \equiv 1$

$$\left(\frac{x-y}{2}\right)^2 + \left(\frac{x+y}{2}\right)^2 = 1$$

$$x^2 - 2xy + y^2 + x^2 + 2xy + y^2 = 4$$

$$2x^2 + 2y^2 = 4$$

$$x^2 + y^2 = 2$$

**11 a** Using the cosine rule

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{8^2 + 12^2 - 10^2}{2 \times 8 \times 12}$$

$$\cos B = \frac{64 + 144 - 100}{192}$$

$$\cos B = \tfrac{108}{192}$$

$$\cos B = \tfrac{9}{16}$$

**b** Since $\sin^2\theta + \cos^2\theta = 1$

$$\sin^2 B + \left(\tfrac{9}{16}\right)^2 = 1$$

$$\sin^2 B = 1 - \tfrac{81}{256}$$

$$= \tfrac{175}{256}$$

So $\sin B = \sqrt{\dfrac{175}{256}}$

$$= \frac{5\sqrt{7}}{16}$$

**12 a** Using the sine rule

$$\frac{\sin Q}{q} = \frac{\sin P}{p}$$

$$\frac{\sin Q}{8} = \frac{\sin 30^\circ}{6}$$

$$\sin Q = \frac{8 \sin 30^\circ}{6}$$

$$= \frac{8 \times \frac{1}{2}}{6}$$

$$= \frac{2}{3}$$

**b** Since $\sin^2\theta + \cos^2\theta = 1$

$$\left(\tfrac{2}{3}\right)^2 + \cos^2 Q = 1$$

$$\cos^2 Q = 1 - \tfrac{4}{9}$$

$$= \tfrac{5}{9}$$

Since $Q$ is obtuse $Q$ is in the second quadrant where cosine is negative.

So $\cos Q = -\dfrac{\sqrt{5}}{3}$