## Algebraic methods, 7 Mixed Exercise

**1 a** $\frac{3x^4-21x}{3x}=\frac{3x^4}{3x}-\frac{21x}{3x}$
$=x^3-7$

**b** $\frac{x^2-2x-24}{x^2-7x+6}$
$=\frac{(x-6)(x+4)}{(x-6)(x-1)}$
$=\frac{x+4}{x-1}$

**c** $\frac{2x^2+7x-4}{2x^2+9x+4}$
$=\frac{(2x-1)(x+4)}{(2x+1)(x+4)}$
$=\frac{2x-1}{2x+1}$

**2**
$$\begin{array}{r}
3x^2+5 \\
x+4\overline{)3x^3+12x^2+5x+20} \\
\underline{3x^3+12x^2} \qquad\quad \\
0+5x+20 \\
\underline{5x+20} \\
0
\end{array}$$

So $\frac{3x^3+12x^2+5x+20}{x+4}=3x^2+5$

**3**
$$\begin{array}{r}
2x^2-2x+5 \\
x+1\overline{)2x^3+0x^2+3x+5} \\
\underline{2x^3+2x^2} \qquad\quad \\
-2x^2+3x \quad \\
\underline{-2x^2-2x} \quad \\
5x+5 \\
\underline{5x+5} \\
0
\end{array}$$

So $\frac{2x^3+3x+5}{x+1}=2x^2-2x+5$

**4 a** $f(x)=2x^3-2x^2-17x+15$
$f(3)=2(3)^3-2(3)^2-17(3)+15$
$=54-18-51+15$
$=0$
So $(x-3)$ is a factor of $2x^3-2x^2-17x+15$.

**b**
$$\begin{array}{r}
2x^2+4x-5 \\
x-3\overline{)2x^3-2x^2-17x+15} \\
\underline{2x^3-6x^2} \qquad\qquad \\
4x^2-17x \quad \\
\underline{4x^2-12x} \quad \\
-5x+15 \\
\underline{-5x+15} \\
0
\end{array}$$

$2x^3-2x^2-17x+15$
$=(x-3)(2x^2+4x-5)$
So $A=2$, $B=4$, $C=-5$

**5 a** $f(x)=x^3+4x^2-3x-18$
$f(2)=(2)^3+4(2)^2-3(2)-18$
$=8+16-6-18$
$=0$
So $(x-2)$ is a factor of $x^3+4x^2-3x-18$.

**b**
$$\begin{array}{r}
x^2+6x+9 \\
x-2\overline{)x^3+4x^2-3x-18} \\
\underline{x^3-2x^2} \qquad\qquad \\
6x^2-3x \quad \\
\underline{6x^2-12x} \quad \\
9x-18 \\
\underline{9x-18} \\
0
\end{array}$$

$x^3+4x^2-3x-18=(x-2)(x^2+6x+9)$
$=(x-2)(x+3)^2$
So $p=1$, $q=3$

**6** $f(x)=2x^3+3x^2-18x+8$
$f(2)=2(2)^3+3(2)^2-18(2)+8$
$=16+12-36+8$
$=0$
So $(x-2)$ is a factor of $2x^3+3x^2-18x+8$.

**6**
$$\begin{array}{r} 2x^2+7x-4 \\ x-2\overline{)2x^3+3x^2-18x+8} \\ \underline{2x^3-4x^2} \qquad\qquad \\ 7x^2-18x \qquad \\ \underline{7x^2-14x} \qquad \\ -4x+8 \\ \underline{-4x+8} \\ 0 \end{array}$$

$2x^3 + 3x^2 - 18x + 8 = (x - 2)(2x^2 + 7x - 4)$
$= (x - 2)(2x - 1)(x + 4)$

**7** $f(x) = x^3 - 3x^2 + kx - 10$
$f(2) = 0$
$(2)^3 - 3(2)^2 + k(2) - 10 = 0$
$8 - 12 + 2k - 10 = 0$
$2k = 14$
$k = 7$

**8 a** $f(x) = 2x^2 + px + q$
$f(-3) = 0$
$2(-3)^2 + p(-3) + q = 0$
$18 - 3p + q = 0$
$3p - q = 18$ **(1)**
$f(4) = 21$
$2(4)^2 + p(4) + q = 21$
$32 + 4p + q = 21$
$4p + q = -11$ **(2)**
**(1)** + **(2)**:
$7p = 7$
$p = 1$
Substituting in **(2)**:
$4(1) + q = -11$
$q = -15$
Checking in (1):
$3p - q = 3(1) - (-15) = 3 + 15 = 18$✓
So $p = 1$, $q = -15$

**b** $f(x) = 2x^2 + x - 15$
$= (2x - 5)(x + 3)$

**9 a** $h(x) = x^3 + 4x^2 + rx + s$
$h(-1) = 0$
$(-1)^3 + 4(-1)^2 + r(-1) + s = 0$
$-1 + 4 - r + s = 0$
$r - s = 3$ **(1)**
$h(2) = 30$
$(2)^3 + 4(2)^2 + r(2) + s = 30$
$8 + 16 + 2r + s = 30$
$2r + s = 6$ **(2)**

**9 a** **(1)** + **(2)**:
$3r = 9$
$r = 3$
Substituting in **(1)**
$3 - s = 3$
$s = 0$
Checking in **(2)**:
$2r + s = 2(3) + (0) = 6$✓
So $r = 3$, $s = 0$

**b** $h(x) = x^3 + 4x^2 + 3x$
$= x(x^2 + 4x + 3)$
$= x(x + 3)(x + 1)$

**10 a** $g(x) = 2x^3 + 9x^2 - 6x - 5$
$g(1) = 2(1)^3 + 9(1)^2 - 6(1) - 5$
$= 2 + 9 - 6 - 5$
$= 0$
So $(x - 1)$ is a factor of $2x^3 + 9x^2 - 6x - 5$.

$$\begin{array}{r} 2x^2+11x+5 \\ x-1\overline{)2x^3+9x^2-6x-5} \\ \underline{2x^3-2x^2} \qquad\qquad \\ 11x^2-6x \qquad \\ \underline{11x^2-11x} \qquad \\ 5x-5 \\ \underline{5x-5} \\ 0 \end{array}$$

$g(x) = 2x^3 + 9x^2 - 6x - 5$
$= (x - 1)(2x^2 + 11x + 5)$
$= (x - 1)(2x + 1)(x + 5)$

**b** $g(x) = 0$
$(x - 1)(2x + 1)(x + 5) = 0$
So $x = 1$, $x = -\frac{1}{2}$ or $x = -5$

**11 a** $f(x) = x^3 + x^2 - 5x - 2$
$f(2) = (2)^3 + (2)^2 - 5(2) - 2$
$= 8 + 4 - 10 - 2$
$= 0$
So $(x - 2)$ is a factor of $x^3 + x^2 - 5x - 2$.

**11 b**

$$\begin{array}{r} x^2+3x+1 \\ x-2\overline{\big)\,x^3+x^2-5x-2} \\ \underline{x^3-2x^2} \\ 3x^2-5x \\ \underline{3x^2-6x} \\ x-2 \\ \underline{x-2} \\ 0 \end{array}$$

$f(x) = x^3 + x^2 - 5x - 2$
$= (x - 2)(x^2 + 3x + 1)$

$f(x) = 0$ when $x = 2$
or $x^2 + 3x + 1 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-3 \pm \sqrt{(3)^2 - 4(1)(1)}}{2(1)}$$

$$= \frac{-3 \pm \sqrt{5}}{2}$$

So the solutions are $x = 2$, $x = \dfrac{-3+\sqrt{5}}{2}$

and $x = \dfrac{-3-\sqrt{5}}{2}$.

**12**

$$\begin{array}{r} 2x^2-7x+3 \\ x+1\overline{\big)\,2x^3-5x^2-4x+3} \\ \underline{2x^3+2x^2} \\ -7x^2-4x \\ \underline{-7x^2-7x} \\ 3x+3 \\ \underline{3x+3} \\ 0 \end{array}$$

$2x^3 - 5x^2 - 4x + 3 = (x + 1)(2x^2 - 7x + 3)$
$= (x + 1)(2x - 1)(x - 3)$

The roots are $x = -1$, $x = \frac{1}{2}$ and $x = 3$.

So the positive roots are $x = \frac{1}{2}$ and $x = 3$.

**13 a** $f(x) = x^3 - 2x^2 - 19x + 20$
$f(-4) = (-4)^3 - 2(-4)^2 - 19(-4) + 20$
$= -64 - 32 + 76 + 20$
$= 0$

The remainder is 0.

**13 b**

$$\begin{array}{r} x^2-6x+5 \\ x+4\overline{\big)\,x^3-2x^2-19x+20} \\ \underline{x^3+4x^2} \\ -6x^2-19x \\ \underline{-6x^2-24x} \\ 5x+20 \\ \underline{5x+20} \\ 0 \end{array}$$

$f(x) = x^3 - 2x^2 - 19x + 20$
$= (x + 4)(x^2 - 6x + 5)$
$= (x + 4)(x - 5)(x - 1)$
$f(x) = 0$ when
$x = -4$, $x = 5$ or $x = 1$

**14 a** $f(x) = 6x^3 + 17x^2 - 5x - 6$

$$f\left(\frac{2}{3}\right) = 6\left(\frac{2}{3}\right)^3 + 17\left(\frac{2}{3}\right)^2 - 5\left(\frac{2}{3}\right) - 6$$

$$= 6\left(\frac{8}{27}\right) + 17\left(\frac{4}{9}\right) - 5\left(\frac{2}{3}\right) - 6$$

$$= \frac{16}{9} + \frac{68}{9} - \frac{10}{3} - 6$$

$= 0$

So $(3x - 2)$ is a factor of $f(x)$.

$$\begin{array}{r} 2x^2+7x+3 \\ 3x-2\overline{\big)\,6x^3+17x^2-5x-6} \\ \underline{6x^3-4x^2} \\ 21x^2-5x \\ \underline{21x^2-14x} \\ 9x-6 \\ \underline{9x-6} \\ 0 \end{array}$$

$f(x) = 6x^3 + 17x^2 - 5x - 6$
$= (3x - 2)(2x^2 + 7x + 3)$

So $a = 2$, $b = 7$, $c = 3$

**b** $f(x) = (3x - 2)(2x^2 + 7x + 3)$
$= (3x - 2)(2x + 1)(x + 3)$

**c** $(3x - 2)(2x + 1)(x + 3) = 0$

The real roots are $x = \frac{2}{3}$, $x = -\frac{1}{2}$ and $x = -3$.

**15** LHS $= \dfrac{x-y}{\left(\sqrt{x}-\sqrt{y}\right)} \times \dfrac{\left(\sqrt{x}+\sqrt{y}\right)}{\left(\sqrt{x}+\sqrt{y}\right)}$

$= \dfrac{(x-y)\left(\sqrt{x}+\sqrt{y}\right)}{x-y}$

$= \sqrt{x}+\sqrt{y}$

= RHS

So $\dfrac{x-y}{\sqrt{x}-\sqrt{y}} \equiv \sqrt{x}+\sqrt{y}$

**16** Completing the square:
$n^2 - 8n + 20 = (n-4)^2 + 4$
The minimum value is 4, so $n^2 - 8n + 20$ is always positive.

**17** $A(1,1)$, $B(3,2)$, $C(4,0)$ and $D(2, -1)$

The gradient of line $AB = \dfrac{2-1}{3-1} = \dfrac{1}{2}$

The gradient of line $BC = \dfrac{0-2}{4-3} = -2$

The gradient of line $CD = \dfrac{-1-0}{2-4} = \dfrac{1}{2}$

The gradient of line $AD = \dfrac{-1-1}{2-1} = -2$

*AB* and *BC*, *BC* and *CD*, *CD* and *AD* and *AB* and *AD* are all perpendicular.

Distance $AB = \sqrt{(3-1)^2 + (2-1)^2}$
$= \sqrt{5}$

Distance $BC = \sqrt{(4-3)^2 + (0-2)^2}$
$= \sqrt{5}$

Distance $CD = \sqrt{(2-4)^2 + (-1-0)^2}$
$= \sqrt{5}$

Distance $AD = \sqrt{(2-1)^2 + (-1-1)^2}$
$= \sqrt{5}$

All four sides are equal and all four angles are right angles, therefore *ABCD* is a square.

**18** 1 + 3 = even
3 + 5 = even
5 + 7 = even
7 + 9 = even
So the sum of two consecutive positive odd numbers is always even.

**19** To show something is untrue you only need to find one counter example.
Example: when $n = 6$,
$n^2 - n + 3 = 6^2 - 6 + 3 = 33$
which is not a prime number.
So the statement is untrue.

**20** LHS $= \left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right)$

$= x^{\frac{7}{3}} + x^{\frac{1}{3}} - x^{\frac{1}{3}} - x^{-\frac{5}{3}}$

$= x^{\frac{7}{3}} - x^{-\frac{5}{3}}$

$= x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$

= RHS

So $\left(x - \dfrac{1}{x}\right)\left(x^{\frac{4}{3}} + x^{-\frac{2}{3}}\right) \equiv x^{\frac{1}{3}}\left(x^2 - \dfrac{1}{x^2}\right)$

**21** Remember, in an identity you can start from the RHS or the LHS. Here it is easier to start from the RHS.
RHS $= (x+4)(x-5)(2x+3)$
$= (x+4)(2x^2 - 7x - 15)$
$= 2x^3 + x^2 - 43x - 60$
= LHS
So $2x^3 + x^2 - 43x - 60$
$\equiv (x+4)(x-5)(2x+3)$

**22** $x^2 - kx + k = 0$ has two equal roots,
so $b^2 - 4ac = 0$
$k^2 - 4k = 0$
$k(k-4) = 0$
$k = 4$ or 0.
So $k = 4$ is a solution.

**23** Using Pythagoras' theorem:
The distance between opposite edges

$= 2\left(\left(\sqrt{3}\right)^2 - \left(\dfrac{\sqrt{3}}{2}\right)^2\right)$

$= 2\left(3 - \dfrac{3}{4}\right)$

$= \dfrac{9}{2}$

$\dfrac{9}{2}$ is rational.

**24 a** Let the first even number be $2n$.
The next even number is $2n + 2$.
$(2n + 2)^2 - (2n)^2 = 4n^2 + 8n + 4 - 4n^2$
$= 8n + 4$
$= 4(2n + 1)$
$4(2n + 1)$ is a multiple of 4 so is always divisible by 4.
So the difference of the squares of two consecutive even numbers is always divisible by 4.

**b** Let the first odd number be $2n - 1$.
The next odd number is $2n + 1$.
$(2n + 1)^2 - (2n - 1)^2$
$= (4n^2 + 4n + 1) - (4n^2 - 4n + 1)$
$= 8n$
$8n$ is a multiple of 8, which is always divisible by 4, so the statement is also true for odd numbers.

**25 a** The assumption is that $x$ is positive.

**b** When $x = 0$, $1 + 0^2 = (1 + 0)^2$

**Challenge**

**1** **a** Diameter of circle = 1,
so side of outside square = 1
Using Pythagoras' theorem:

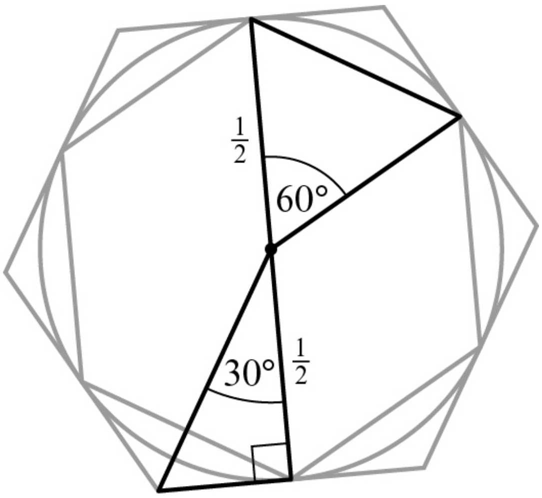
Perimeter of the inside square = $4\left(\sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\right)=\frac{4}{\sqrt{2}}=2\sqrt{2}$

Perimeter of the outside square = 4 × 1 = 4
The circumference of the circle is between the perimeters of the two squares, so $2\sqrt{2} < \pi < 4$.

**b** Perimeter of inside hexagon = $6\times\frac{1}{2}=3$ because the triangles with 60° angles are equilateral.

Perimeter of outside hexagon = $6\times\frac{\sqrt{3}}{3}=2\sqrt{3}$

The circumference of the circle is between the perimeters of the two hexagons, so $3 < \pi < 2\sqrt{3}$




**2**

$$
\begin{array}{r}
ax^2+(b+ap)x+(c+bp+ap^2) \\
x-p\,\big)\,ax^3+bx^2\qquad+cx\qquad+d \\
ax^3-apx^2 \\
(b+ap)x^2+cx \\
(b+ap)x^2-(bp+ap^2)x \\
(c+bp+ap^2)x+d \\
(c+bp+ap^2)x-(cp+bp^2+ap^3) \\
d+cp+bp^2+ap^3
\end{array}
$$

So $\frac{ax^3+bx^2+cx+d}{x-p}=ax^2+(b+ap)x+(c+bp+ap^2)$ with remainder.
So, $d+cp+bp^2+ap^3$
$f(p)=ap^3+bp^2+cp+d=0$, which matches the remainder $d+cp+bp^2+ap^3=0$
Therefore $(x-p)$ is a factor of $f(x)$.