Algebraic methods 7E

1 Example: when $n = 1$, $m = 3$
and 3 is not divisible by 10.
So the statement is not true.

2 3, 5, 7, 11, 13, 17, 19, 23 are the prime numbers between 2 and 26.
The other odd numbers between 2 and 26 are 9, 15, 21, 25.
$9 = 3 \times 3$
$15 = 5 \times 3$
$21 = 7 \times 3$
$25 = 5 \times 5$
So every odd integer between 2 and 26 is either prime or the product of two primes.

3 $1^2 + 2^2 = \text{odd}$
$2^2 + 3^2 = \text{odd}$
$3^2 + 4^2 = \text{odd}$
$4^2 + 5^2 = \text{odd}$
$5^2 + 6^2 = \text{odd}$
$6^2 + 7^2 = \text{odd}$
$7^2 + 8^2 = \text{odd}$
So the sum of two consecutive square numbers between 1^2 and 8^2 is always an odd number.

4 Break down the integers into numbers divisible by 3 and numbers giving a remainder of 1 or 2 when divided by 3.

$(3n)^3 = 27n^3 = 9n(3n^2)$ which is a multiple of 9.

$$(3n + 1)^3 = 27n^3 + 27n^2 + 9n + 1 = 9n(3n^2 + 3n + 1) + 1$$
which is one more than a multiple of 9.

$$(3n + 2)^3 = 27n^3 + 54n^2 + 36n + 8 = 9n(3n^2 + 6n + 4) + 8$$
which is one less than a multiple of 9.

So all cube numbers are either a multiple of 9 or 1 more or 1 less than a multiple of 9.

5 a Example: when $n = 2$, $2^4 - 2 = 14$
14 is not divisible by 4.

b Any square number has an odd number of factors, for example 25 has 3 factors.

c Example: when $n = \frac{1}{2}$,
$$2\left(\frac{1}{2}\right)^2 - 6\left(\frac{1}{2}\right) + 1 = 2\left(\frac{1}{4}\right) - 3 + 1 = \frac{1}{2} - 2 = -\frac{3}{2}$$
which is negative.

d Example: when $n = 1$,
$2(1)^2 - 2(1) - 4 = 2 - 2 - 4 = -4$
which is not a multiple of 3.

6 a The error lies in the last stage. We can only write this statement if $3(x^2)y + 3x(y^2)$ is greater than zero. No work has been done to prove or disprove this.

b Example, when $x = 0$ and $y = 0$,
$0^3 + 0^3 = (0 + 0)^3$

7 $(x + 5)^2 \geq 0$ for all real values of x
As $(x + 5)^2 = x^2 + 10x + 25$
and $(x + 6)^2 = x^2 + 12x + 36$
$(x + 5)^2 + 2x + 11 = (x + 6)^2$
So $(x + 6)^2 \geq 2x + 11$

8 As a is positive, multiplying both sides by a does not reverse the inequality
So $a^2 + 1 \geq 2a$
Then $a^2 - 2a + 1 \geq 0$
Factorising gives
$(a - 1)^2 \geq 0$ which we know is true.

9 a By squaring both sides, consider $(p + q)^2$
$$(p + q)^2 = p^2 + 2pq + q^2 = (p - q)^2 + 4pq$$
$(p - q)^2 \geq 0$ since it is a square
so $(p + q)^2 \geq 4pq$
p and q are both positive
so $p > 0$ and $q > 0$
Therefore, $p + q > 0$
So $p + q \geq \sqrt{4pq}$

9 b When $p = q = -1$, $p + q = -2$
and $\sqrt{4pq} = 2$
but $-2 < 2$, i.e $p + q < \sqrt{4pq}$
which is inconsistent.

10 a The student had forgotten the significance of x and y both being negative i.e the left hand side is negative while the right hand side can be positive. In this case the inequality could not be true.

10 b When $x = y = -1$, $x + y = -2$
and $\sqrt{x^2 + y^2} = \sqrt{2}$
$-2 < \sqrt{2}$

c $(x + y)^2 = x^2 + 2xy + y^2$
As $x > 0$ and $y > 0$ then $2xy > 0$.
So $x^2 + 2xy + y^2 \geq x^2 + y^2$
As $x + y > 0$, square root both sides
$x + y \geq \sqrt{x^2 + y^2}$