Algebraic methods 7D

1 $n^2 - n = n(n - 1)$
If n is even, $n - 1$ is odd
and even × odd = even
If n is odd, $n - 1$ is even
and odd × even = even
So $n^2 - n$ is even for all values of n.

2 LHS $= \dfrac{x}{(1+\sqrt{2})} \times \dfrac{(1-\sqrt{2})}{(1-\sqrt{2})}$

$= \dfrac{x(1-\sqrt{2})}{(1-2)}$

$= \dfrac{x - x\sqrt{2}}{-1}$

$= x\sqrt{2} - x$

= RHS

So $\dfrac{x}{(1+\sqrt{2})} \equiv x\sqrt{2} - x$

3 LHS $= (x+\sqrt{y})(x-\sqrt{y})$

$= x^2 - x\sqrt{y} + x\sqrt{y} - y$

$= x^2 - y$

= RHS

So $(x+\sqrt{y})(x-\sqrt{y}) \equiv x^2 - y$

4 LHS $= (2x - 1)(x + 6)(x - 5)$
$= (2x - 1)(x^2 + x - 30)$
$= 2x^3 + x^2 - 61x + 30$
= RHS
So $(2x - 1)(x + 6)(x - 5) \equiv 2x^3 + x^2 - 61x + 30$

5 Completing the square:

$x^2 + bx = \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

So $x^2 + bx \equiv \left(x + \dfrac{b}{2}\right)^2 - \left(\dfrac{b}{2}\right)^2$

6 $x^2 + 2bx + c = 0$
Completing the square:
$(x + b)^2 - b^2 + c = 0$
$(x + b)^2 = b^2 - c$
$x + b = \pm\sqrt{b^2 - c}$
$x = -b \pm \sqrt{b^2 - c}$
So the solutions of $x^2 + 2bx + c = 0$ are $x = -b \pm \sqrt{b^2 - c}$.

7 LHS $= \left(x - \dfrac{2}{x}\right)^3$

$= \left(x - \dfrac{2}{x}\right)\left(x^2 - 4 + \dfrac{4}{x^2}\right)$

$= x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

= RHS

So $\left(x - \dfrac{2}{x}\right)^3 \equiv x^3 - 6x + \dfrac{12}{x} - \dfrac{8}{x^3}$

8 LHS $= \left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right)$

$= x^{\frac{9}{2}} + x^{\frac{1}{2}} - x^{\frac{1}{2}} - x^{-\frac{7}{2}}$

$= x^{\frac{9}{2}} - x^{-\frac{7}{2}}$

$= x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

= RHS

So $\left(x^3 - \dfrac{1}{x}\right)\left(x^{\frac{3}{2}} + x^{-\frac{5}{2}}\right) \equiv x^{\frac{1}{2}}\left(x^4 - \dfrac{1}{x^4}\right)$

9 $3n^2 - 4n + 10 = 3\left(n^2 - \dfrac{4}{3}n + \dfrac{10}{3}\right)$

$= 3\left(\left(n - \dfrac{2}{3}\right)^2 - \dfrac{4}{9} + \dfrac{10}{3}\right)$

$= 3\left(n - \dfrac{2}{3}\right)^2 + \dfrac{26}{3}$

The minimum value is $\frac{26}{3}$ so
$3n^2 - 4n + 10$ is always positive.

10 $-n^2 - 2n - 3 = -(n^2 + 2n + 3)$
$= -((n + 1)^2 - 1 + 3)$
$= -(n + 1)^2 - 2$
The maximum value is -2,
so $-n^2 - 2n - 3$ is always negative.

11 $x^2 + 8x + 20$
Complete the square
$(x + 4)^2 - 16 + 20 = (x + 4)^2 + 4$

11 The minimum value of $(x + 4)^2 + 4$ is 4
So $(x + 4)^2 + 4 \geq 4$
Therefore, $x^2 + 8x + 20 \geq 4$

12 $kx^2 + 5kx + 3 = 0$ has no real roots,
so $b^2 - 4ac < 0$
$(5k)^2 - 4k(3) < 0$
$25k^2 - 12k < 0$
$k(25k - 12) < 0$
$0 < k < \frac{12}{25}$
When $k = 0$:
$(0)x^2 + 5(0)x + 3 = 0$
$3 = 0$
which is impossible, so no real roots.
So combining these:
$0 \leq k < \frac{12}{25}$

13 $px^2 - 5x - 6 = 0$ has two distinct real roots, so
$b^2 - 4ac > 0$
$25 + 24p > 0$
$p > -\frac{25}{24}$

14 $A(1, 2)$, $B(1, 2)$ and $C(2, 4)$
The gradient of line $AB = \frac{2-1}{1-3} = -\frac{1}{2}$
The gradient of line $BC = \frac{4-2}{2-1} = 2$
The gradient of line $AC = \frac{4-1}{2-3} = -3$
The gradients are different so the three points are not collinear.
Hence ABC is a triangle.
Gradient of AB × gradient of BC
$= -\frac{1}{2} \times 2$
$= -1$
So AB is perpendicular to BC,
and the triangle is a right-angled triangle.

15 $A(1, 1)$, $B(2, 4)$, $C(6, 5)$ and $D(5, 2)$
The gradient of line $AB = \frac{4-1}{2-1} = 3$
The gradient of line $BC = \frac{5-4}{6-2} = \frac{1}{4}$
The gradient of line $CD = \frac{2-5}{5-6} = 3$
The gradient of line $AD = \frac{2-1}{5-1} = \frac{1}{4}$

15 Gradient of AB = gradient of CD, so AB and CD are parallel.
Gradient of BC = gradient of AD, so BC and AD are parallel.

So $ABCD$ can be a parallelogram or a rectangle and we need to check further.
Since there is not a pair of gradients which multiply to give −1 there is no right angle.
Hence $ABCD$ is a parallelogram.

16 $A(2, 1)$, $B(5, 2)$, $C(4, -1)$ and $D(1,-2)$

The gradient of line $AB = \frac{2-1}{5-2} = \frac{1}{3}$
The gradient of line $BC = \frac{-1-2}{4-5} = 3$
The gradient of line $CD = \frac{-2+1}{1-4} = \frac{1}{3}$
The gradient of line $AD = \frac{-2-1}{1-2} = 3$

Gradient of AB = gradient of CD,
so AB and CD are parallel.
Gradient of BC = gradient of AD,
so BC and AD are parallel.

Distance $AB = \sqrt{(5-2)^2 + (2-1)^2}$
$= \sqrt{10}$

Distance $BC = \sqrt{(4-5)^2 + (-1-2)^2}$
$= \sqrt{10}$

Distance $CD = \sqrt{(1-4)^2 + (-2+1)^2}$
$= \sqrt{10}$

Distance $AD = \sqrt{(1-2)^2 + (-2-1)^2}$
$= \sqrt{10}$

All four sides are equal. Since no pairs of gradients multiply to give −1 there are no right angles at a vertex so this is not a square. Hence $ABCD$ is a rhombus.

17 $A(-5,2)$, $B(-3,-4)$ and $C(3,-2)$

The gradient of line $AB = \frac{-4-2}{-3+5} = -3$

17 The gradient of line $BC = \frac{-2+4}{3+3} = \frac{1}{3}$

The gradient of line $AC = \frac{-2-2}{3+5} = -\frac{1}{2}$

The gradients are different so the three points are not collinear. Hence ABC is a triangle.

Gradient of AB × gradient of BC

$= -3 \times \frac{1}{3}$

$= -1$

So AB is perpendicular to BC.

Distance $AB = \sqrt{(-3+5)^2 + (-4-2)^2}$

$= \sqrt{40}$

Distance $BC = \sqrt{(3+3)^2 + (-2+4)^2}$

$= \sqrt{40}$

$AB = BC$

As two sides are equal and an angle is right-angled, ABC is an isosceles right-angled triangle.

18 Substituting $y = ax$ into $(x-1)^2 + y^2 = k$:

$(x-1)^2 + a^2x^2 = k$

$x^2 - 2x + 1 + a^2x^2 - k = 0$

$x^2(1 + a^2) - 2x + 1 - k = 0$

The straight line cuts the circle at two distinct points, so this equation has two distinct real roots, so

$b^2 - 4ac > 0$

$(-2)^2 - 4(1 + a^2)(1 - k) > 0$

$4 - 4(1 - k + a^2 - ka^2) > 0$

$4k - 4a^2 + 4ka^2 > 0$

$-a^2 + k + ka^2 > 0$

$-a^2 + k(1 + a^2) > 0$

$k > \frac{a^2}{1+a^2}$

19 $4y - 3x + 26 = 0$

$4y = 3x - 26$

$y = \frac{3}{4}x - \frac{13}{2}$

Substituting $y = \frac{3}{4}x - \frac{13}{2}$ into $(x + 4)^2 + (y - 3)^2 = 100$:

19 $(x + 4)^2 + \left(\frac{3}{4}x - \frac{19}{2}\right)^2 = 100$

$x^2 + 8x + 16 + \frac{9}{16}x^2 - \frac{57}{4}x + \frac{361}{4} - 100 = 0$

$16x^2 + 128x + 256 + 9x^2 - 228x + 1444 - 1600 = 0$

$25x^2 - 100x + 100 = 0$

$x^2 - 4x + 4 = 0$

$(x - 2)^2 = 0$

$x = 2$

There is only one solution so the line $4y - 3x + 26 = 0$ only touches the circle in one place, so it is a tangent to the circle.

20 Area of square $= (a + b)^2 = a^2 + 2ab + b^2$

Shaded area $= 4\left(\frac{1}{2}ab\right) = 2ab$

Area of smaller square

$= a^2 + 2ab + b^2 - 2ab$

$= a^2 + b^2$

$= c^2$

Challenge

1 Find the equations of the perpendicular bisectors to the chords AB and BC:

$A(7, 8)$ and $B(-1, 8)$

Midpoint $= \left(\frac{7-1}{2}, \frac{8+8}{2}\right) = (3, 8)$

The gradient of the line segment AB

$= \frac{8-8}{-1-7}$

$= 0$

So the line perpendicular to AB is a vertical line $x = 3$.

$B(-1, 8)$ and $C(6, 1)$

Midpoint $= \left(\frac{-1+6}{2}, \frac{8+1}{2}\right)$

$= \left(\frac{5}{2}, \frac{9}{2}\right)$

The gradient of the line segment BC

$= \frac{1-8}{6+1}$

$= -1$

Challenge

1 So the gradient of the line perpendicular to BC is 1.

The equation of the perpendicular line is
$y - y_1 = m(x - x_1)$
$m = 1$ and $(x_1, y_1) = \left(\frac{5}{2}, \frac{9}{2}\right)$

So $y - \frac{9}{2} = x - \frac{5}{2}$
$y = x + 2$

AB and BC intersect at the centre of the circle, so solving $x = 3$ and $y = x + 2$ simultaneously:
$x = 3, y = 5$
Centre of the circle, X, is (3, 5).

Distance $AX = \sqrt{(7-3)^2 + (8-5)^2}$
$= \sqrt{25}$
$= 5$

Distance $BX = \sqrt{(-1-3)^2 + (8-5)^2}$
$= \sqrt{25}$
$= 5$

Distance $CX = \sqrt{(6-3)^2 + (1-5)^2}$
$= \sqrt{25}$
$= 5$

Distance $DX = \sqrt{(0-3)^2 + (9-5)^2}$
$= \sqrt{25}$
$= 5$

The distance from the centre of the circle to all four points is 5 units, so all four points lie on a circle with centre (3, 5).

2 $3 = 2^2 - 1^2$
$5 = 3^2 - 2^2$
$7 = 4^2 - 3^2$
$11 = 6^2 - 5^2$

Let p be a prime number greater than 2.

$\left(\frac{1}{2}(p+1)\right)^2 - \left(\frac{1}{2}(p-1)\right)^2$
$= \frac{1}{4}\left((p+1)^2 - (p-1)^2\right)$
$= \frac{1}{4}(4p)$
$= p$

So any odd prime number can be written as the difference of two squares.