Algebraic methods 7C

1 a $f(x) = 4x^3 - 3x^2 - 1$
$f(1) = 4(1)^3 - 3(1)^2 - 1$
$= 4 - 3 - 1$
$= 0$
So $(x - 1)$ is a factor of $4x^3 - 3x^2 - 1$.

b $f(x) = 5x^4 - 45x^2 - 6x - 18$
$f(-3) = 5(-3)^4 - 45(-3)^2 - 6(-3) - 18$
$= 5(81) - 45(9) + 18 - 18$
$= 405 - 405$
$= 0$
So $(x + 3)$ is a factor of
$5x^4 - 45x^2 - 6x - 18$.

c $f(x) = -3x^3 + 13x^2 - 6x + 8$
$f(4) = -3(4)^3 + 13(4)^2 - 6(4) + 8$
$= -192 + 208 - 24 + 8$
$= 0$
So $(x - 4)$ is a factor of
$-3x^3 + 13x^2 - 6x + 8$.

2 $f(x) = x^3 + 6x^2 + 5x - 12$
$f(-1) = (1)^3 + 6(1)^2 + 5(1) - 12$
$= 1 + 6 + 5 - 12$
$= 0$
So $(x - 1)$ is a factor of $x^3 + 6x^2 + 5x - 12$.

$$
\begin{array}{r}
x^2 + 7x + 12 \\
x - 1 \overline{\big) x^3 + 6x^2 + 5x - 12} \\
\underline{x^3 - x^2} \\
7x^2 + 5x \\
\underline{7x^2 - 7x} \\
12x - 12 \\
\underline{12x - 12} \\
0
\end{array}
$$

$x^3 + 6x^2 + 5x - 12 = (x - 1)(x^2 + 7x + 12)$
$= (x - 1)(x + 3)(x + 4)$

3 $f(x) = x^3 + 3x^2 - 33x - 35$
$f(-1) = (-1)^3 + 3(-1)^2 - 33(-1) - 35$
$= -1 + 3 + 33 - 35$
$= 0$
So $(x + 1)$ is a factor of
$x^3 + 3x^2 - 33x - 35$.

3

$$
\begin{array}{r}
x^2 + 2x - 35 \\
x + 1 \overline{\big) x^3 + 3x^2 - 33x - 35} \\
\underline{x^3 + x^2} \\
2x^2 - 33x \\
\underline{2x^2 + 2x} \\
-35x - 35 \\
\underline{-35x - 35} \\
0
\end{array}
$$

$x^3 + 3x^2 - 33x - 35 = (x + 1)(x^2 + 2x - 35)$
$= (x + 1)(x + 7)(x - 5)$

4 $f(x) = x^3 + 7x^2 + 2x + 40$
$f(5) = (5)^3 + 7(5)^2 + 2(5) + 40$
$= 125 - 175 + 10 + 40$
$= 0$
So $(x - 5)$ is a factor of $x^3 + 7x^2 + 2x + 40$.

$$
\begin{array}{r}
x^2 - 2x - 8 \\
x - 5 \overline{\big) x^3 - 7x^2 + 2x + 40} \\
\underline{x^3 - 5x^2} \\
-2x^2 + 2x \\
\underline{-2x^2 + 10x} \\
-8x + 40 \\
\underline{-8x + 40} \\
0
\end{array}
$$

$x^3 - 7x^2 + 2x + 40 = (x - 5)(x^2 - 2x - 8)$
$= (x - 5)(x - 4)(x + 2)$

5 $f(x) = 2x^3 + 3x^2 - 18x + 8$
$f(2) = 2(2)^3 + 3(2)^2 - 18(2) + 8$
$= 16 + 12 - 36 + 8$
$= 0$
So $(x - 2)$ is a factor of
$2x^3 + 3x^2 - 18x + 8$.

5

$$\begin{array}{r l} & 2x^2+7x-4 \\ x-2 & \overline{)\,2x^3+3x^2-18x+8} \\ & \underline{2x^3-4x^2} \\ & 7x^2-18x \\ & \underline{7x^2-14x} \\ & -4x+8 \\ & \underline{-4x+8} \\ & 0 \end{array}$$

$2x^3 + 3x^2 - 18x + 8$
$= (x - 2)(2x^2 + 7x - 4)$
$= (x - 2)(2x - 1)(x + 4)$

6 a $f(x) = x^3 - 10x^2 + 19x + 30$
$f(-1) = (-1)^3 - 10(-1)^2 + 19(-1) + 30$
$= -1 - 10 - 19 + 30$
So $(x + 1)$ is a factor of $x^3 - 10x^2 + 19x + 30$.

$$\begin{array}{r l} & x^2-11x+30 \\ x+1 & \overline{)\,x^3-10x^2+19x+30} \\ & \underline{x^3+x^2} \\ & -11x^2+19x \\ & \underline{-11x^2-11x} \\ & 30x+30 \\ & \underline{30x+30} \\ & 0 \end{array}$$

$x^3 - 10x^2 + 19x + 30$
$= (x + 1)(x^2 - 11x + 30)$
$= (x + 1)(x - 5)(x - 6)$

b $f(x) = x^3 + x^2 - 4x - 4$
$f(-1) = (-1)^3 + (-1)^2 - 4(-1) - 4$
$= -1 + 1 + 4 - 4$
$= 0$
So $(x + 1)$ is a factor of $x^3 + x^2 - 4x - 4$.

$$\begin{array}{r l} & x^2-4 \\ x+1 & \overline{)\,x^3+x^2-4x-4} \\ & \underline{x^3+x^2} \\ & 0 \quad -4x-4 \\ & \underline{-4x-4} \\ & 0 \end{array}$$

$x^3 + x^2 - 4x - 4 = (x + 1)(x^2 - 4)$
$= (x + 1)(x - 2)(x + 2)$

6 c $f(x) = x^3 - 4x^2 - 11x + 30$
$f(2) = (2)^3 - 4(2)^2 - 11(2) + 30$
$= 8 - 16 - 22 + 30$
$= 0$
So $(x - 2)$ is a factor of $x^3 - 4x^2 - 11x + 30$.

$$\begin{array}{r l} & x^2-2x-15 \\ x-2 & \overline{)\,x^3-4x^2-11x+30} \\ & \underline{x^3-2x^2} \\ & -2x^2-11x \\ & \underline{-2x^2+4x} \\ & -15x+30 \\ & \underline{-15x+30} \\ & 0 \end{array}$$

$x^3 - 4x^2 - 11x + 30 = (x - 2)(x^2 - 2x - 15)$
$= (x - 2)(x + 3)(x - 5)$

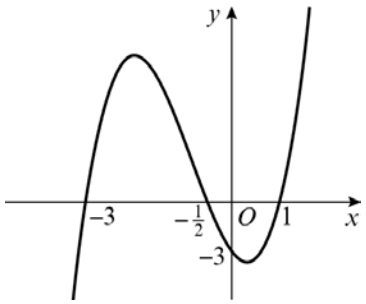
7 a i $f(x) = 2x^3 + 5x^2 - 4x - 3$
$f(1) = 2(1)^3 + 5(1)^2 - 4(1) - 3$
$= 2 + 5 - 4 - 3$
$= 0$
So $(x - 1)$ is a factor of $2x^3 + 5x^2 - 4x - 3$.

$$\begin{array}{r l} & 2x^2+7x+3 \\ x-1 & \overline{)\,2x^3+5x^2-4x-3} \\ & \underline{2x^3-2x^2} \\ & 7x^2-4x \\ & \underline{7x^2-7x} \\ & 3x-3 \\ & \underline{3x-3} \\ & 0 \end{array}$$

$y = 2x^3 + 5x^2 - 4x - 3$
$= (x - 1)(2x^2 + 7x + 3)$
$= (x - 1)(2x + 1)(x + 3)$

ii $0 = (x - 1)(2x + 1)(x + 3)$
So the curve crosses the x-axis at $(1, 0)$, $(-\frac{1}{2}, 0)$ and $(-3, 0)$.
When $x = 0$, $y = (-1)(1)(3) = -3$
The curve crosses the y-axis at $(0, -3)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

7 a ii



b i $f(x) = 2x^3 - 17x^2 + 38x - 15$

$f(3) = 2(3)^3 - 17(3)^2 + 38(3) - 15$
$= 54 - 153 + 114 - 15$
$= 0$

So $(x - 3)$ is a factor of $2x^3 - 17x^2 + 38x - 15$.

$$
\begin{array}{r}
2x^2 - 11x + 5 \\
x - 3 \overline{)\, 2x^3 - 17x^2 + 38x - 15} \\
\underline{2x^3 - 6x^2} \qquad\qquad \\
-11x^2 + 38x \qquad \\
\underline{-11x^2 + 33x} \qquad \\
5x - 15 \\
\underline{5x - 15} \\
0
\end{array}
$$

$y = 2x^3 - 17x^2 + 38x - 15$
$= (x - 3)(2x^2 - 11x + 5)$
$= (x - 3)(2x - 1)(x - 5)$

ii $0 = (x - 3)(2x - 1)(x - 5)$

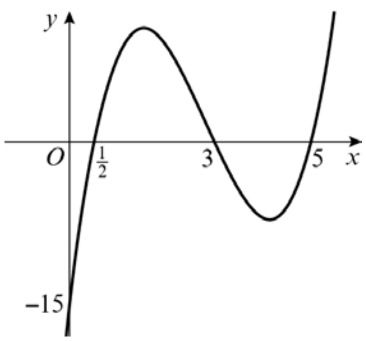
So the curve crosses the x-axis at (3, 0), $(\frac{1}{2}, 0)$ and (5, 0).

When $x = 0$, $y = (-3)(-1)(-5) = -15$

The curve crosses the y-axis at (0, −15).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



c i $f(x) = 3x^3 + 8x^2 + 3x - 2$

$f(-1) = 3(-1)^3 + 8(-1)^2 + 3(-1) - 2$
$= -3 + 8 - 3 - 2$
$= 0$

So $(x + 1)$ is a factor of $3x^3 + 8x^2 + 3x - 2$.

$$
\begin{array}{r}
3x^2 + 5x - 2 \\
x + 1 \overline{)\, 3x^3 + 8x^2 + 3x - 2} \\
\underline{3x^3 + 3x^2} \qquad\qquad \\
5x^2 + 3x \qquad \\
\underline{5x^2 + 5x} \qquad \\
-2x - 2 \\
\underline{-2x - 2} \\
0
\end{array}
$$

$y = 3x^3 + 8x^2 + 3x - 2$
$= (x + 1)(3x^2 + 5x - 2)$
$= (x + 1)(3x - 1)(x + 2)$

ii $0 = (x + 1)(3x - 1)(x + 2)$

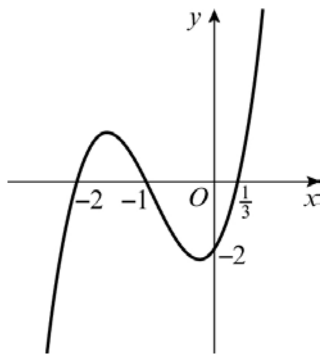
So the curve crosses the x-axis at (−1, 0), $(\frac{1}{3}, 0)$ and (−2, 0).

When $x = 0$, $y = (1)(-1)(2) = -2$

The curve crosses the y-axis at (0, −2).

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



d i $f(x) = 6x^3 + 11x^2 - 3x - 2$

$f(-2) = 6(-2)^3 + 11(-2)^2 - 3(-2) - 2$
$= -48 + 44 + 6 - 2$
$= 0$

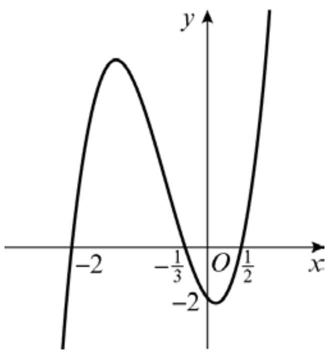
So $(x + 2)$ is a factor of $6x^3 + 11x^2 - 3x - 2$.

7 d i

$$\begin{array}{r} 6x^2 - x - 1 \\ x+2 \overline{)\, 6x^3 + 11x^2 - 3x - 2} \\ \underline{6x^3 + 12x^2} \\ -x^2 - 3x \\ \underline{-x^2 - 2x} \\ -x - 2 \\ \underline{-x - 2} \\ 0 \end{array}$$

$y = 6x^3 + 11x^2 - 3x - 2$
$= (x + 2)(6x^2 - x - 1)$
$= (x + 2)(3x + 1)(2x - 1)$

ii $0 = (x + 2)(3x + 1)(2x - 1)$
So the curve crosses the x-axis at $(-2, 0)$, $(-\frac{1}{3}, 0)$ and $(\frac{1}{2}, 0)$.
When $x = 0$, $y = (2)(1)(-1) = -2$
The curve crosses the y-axis at $(0, -2)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

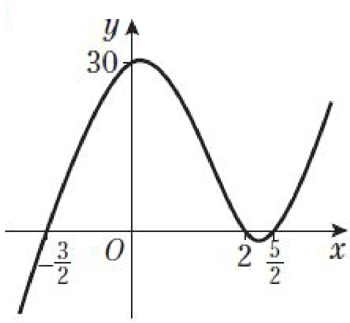


e i $f(x) = 4x^3 - 12x^2 - 7x + 30$
$f(2) = 4(2)^3 - 12(2)^2 - 7(2) + 30$
$= 32 - 48 - 14 + 30$
$= 0$
So $(x - 2)$ is a factor of $4x^3 - 12x^2 - 7x + 30$.

$$\begin{array}{r} 4x^2 - 4x - 15 \\ x-2 \overline{)\, 4x^3 - 12x^2 - 7x + 30} \\ \underline{4x^3 - 8x^2} \\ -4x^2 - 7x \\ \underline{-4x^2 + 8x} \\ -15x + 30 \\ \underline{-15x + 30} \\ 0 \end{array}$$

e i $y = 4x^3 - 12x^2 - 7x + 30$
$= (x - 2)(4x^2 - 4x - 15)$
$= (x - 2)(2x + 3)(2x - 5)$

ii $0 = (x - 2)(2x + 3)(2x - 5)$
So the curve crosses the x-axis at $(2, 0)$, $(-\frac{3}{2}, 0)$ and $(\frac{5}{2}, 0)$.
When $x = 0$, $y = (-2)(3)(-5) = 30$
The curve crosses the y-axis at $(0, 30)$.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



8 $f(x) = 5x^3 - 9x^2 + 2x + a$
$f(1) = 0$
$5(1)^3 - 9(1)^2 + 2(1) + a = 0$
$5 - 9 + 2 + a = 0$
$a = 2$

9 $f(x) = 6x^3 - bx^2 + 18$
$f(-3) = 0$
$6(-3)^3 - b(-3)^2 + 18 = 0$
$-162 - 9b + 18 = 0$
$9b = -144$
$b = -16$

10 $f(x) = px^3 + qx^2 - 3x - 7$
$f(1) = 0$
$p(1)^3 + q(1)^2 - 3(1) - 7 = 0$
$p + q - 3 - 7 = 0$
$p + q = 10$ **(1)**

$f(-1) = 0$
$p(-1)^3 + q(-1)^2 - 3(-1) - 7 = 0$
$-p + q + 3 - 7 = 0$
$-p + q = 4$ **(2)**

(1) + **(2)**:
$2q = 14$
$q = 7$
Substituting in **(1)**:
$p + 7 = 10$
$p = 3$
So $p = 3$, $q = 7$

11 $f(x) = cx^3 + dx^2 - 9x - 10$
$f(-1) = 0$
$c(-1)^3 + d(-1)^2 - 9(-1) - 10 = 0$
$-c + d + 9 - 10 = 0$
$d = c + 1$ **(1)**

$f(2) = 0$
$c(2)^3 + d(2)^2 - 9(2) - 10 = 0$
$8c + 4d - 18 - 10 = 0$
$8c + 4d - 28 = 0$
$8c + 4d = 28$ **(2)**

Substituting (**1**) in (**2**):
$8c + 4(c + 1) = 28$
$12c + 4 = 28$
$c = 2$
Substituting in (**1**):
$d = 2 + 1 = 3$
So $c = 2$, $d = 3$

12 $f(x) = gx^3 + hx^2 - 14x + 24$
$f(-2) = 0$
$g(-2)^3 + h(-2)^2 - 14(-2) + 24 = 0$
$-8g + 4h + 28 + 24 = 0$
$-8g + 4h + 52 = 0$
$h = 2g - 13$ **(1)**

$f(3) = 0$
$g(3)^3 + h(3)^2 - 14(3) + 24 = 0$
$27g + 9h - 42 + 24 = 0$
$27g + 9h = 18$ **(2)**

Substituting (**1**) in (**2**):
$27g + 9(2g - 13) = 18$
$45g = 135$
$g = 3$
Substituting in (**1**):
$h = 2(3) - 13 = -7$
So $g = 3$, $h = -7$

13 a $f(x) = 3x^3 - 12x^2 + 6x - 24$
$f(4) = 3(4)^3 - 12(4)^2 + 6(4) - 24$
$= 192 - 192 + 24 - 24$
$= 0$
So $(x - 4)$ is a factor of $f(x)$.

b
$$\begin{array}{r} 3x^2 + 6 \\ x-4\overline{)3x^3 - 12x^2 + 6x - 24} \\ \underline{3x^3 - 12x^2} \\ 0 + 6x - 24 \\ \underline{6x - 24} \\ 0 \end{array}$$

13 b $f(x) = (x - 4)(3x^2 + 6)$
$(x - 4)(3x^2 + 6) = 0$
Using the discriminant for $3x^2 + 6$:
$b^2 - 4ac = 0 - 4(3)(6) = -72 < 0$.
Therefore $3x^2 + 6$ has no real roots, so $f(x)$ only has one real root of $x = 4$.

14 a $f(x) = 4x^3 + 4x^2 - 11x - 6$
$f(-2) = 4(-2)^3 + 4(-2)^2 - 11(-2) - 6$
$= -32 + 16 + 22 - 6$
$= 0$
So $(x + 2)$ is a factor of $f(x)$.

b
$$\begin{array}{r} 4x^2 - 4x - 3 \\ x+2\overline{)4x^3 + 4x^2 - 11x - 6} \\ \underline{4x^3 + 8x^2} \\ -4x^2 - 11x \\ \underline{-4x^2 - 8x} \\ -3x - 6 \\ \underline{-3x - 6} \\ 0 \end{array}$$

$f(x) = (x + 2)(4x^2 - 4x - 3)$
$= (x + 2)(2x - 3)(2x + 1)$

c $0 = (x + 2)(2x - 3)(2x + 1)$
The solutions are $x = -2$, $x = \frac{3}{2}$ and $x = -\frac{1}{2}$.

15 a $f(x) = 9x^4 - 18x^3 - x^2 + 2x$
$f(2) = 9(2)^4 - 18(2)^3 - (2)^2 + 2(2)$
$= 144 - 144 - 4 + 4$
$= 0$
So $(x - 2)$ is a factor of $9x^4 - 18x^3 - x^2 + 2x$.

b
$$\begin{array}{r} 9x^3 - x \\ x-2\overline{)9x^4 - 18x^3 - x^2 + 2x} \\ \underline{9x^4 - 18x^3} \\ 0 - x^2 + 2x \\ \underline{-x^2 + 2x} \\ 0 \end{array}$$

$9x^4 - 18x^3 - x^2 + 2x$
$= (x - 2)(9x^3 - x)$
$= x(x - 2)(9x^2 - 1)$
$= x(x - 2)(3x + 1)(3x - 1)$
$0 = x(x - 2)(3x + 1)(3x - 1)$

15 b The solutions are $x = 0$, $x = 2$, $x = -\frac{1}{3}$ and $x = \frac{1}{3}$.

Challenge

a $f(x) = 2x^4 - 5x^3 - 42x^2 - 9x + 54$
$f(1) = 2(1)^4 - 5(1)^3 - 42(1)^2 - 9(1) + 54$
$= 0$
$f(-3) = 2(-3)^4 - 5(-3)^3 - 42(-3)^2 - 9(-3) + 54$
$= 162 + 135 - 378 + 27 + 54$
$= 0$

b

$$
\begin{array}{r}
2x^3 - 3x^2 - 45x - 54 \\
x-1 \overline{\big) 2x^4 - 5x^3 - 42x^2 - 9x + 54} \\
\underline{2x^4 - 2x^3} \qquad\qquad\qquad\quad \\
-3x^3 - 42x^2 \qquad\qquad \\
\underline{-3x^3 + 3x^2} \qquad\qquad \\
-45x^2 - 9x \qquad \\
\underline{-45x^2 + 45x} \qquad \\
-54x + 54 \\
\underline{-54x + 54} \\
0
\end{array}
$$

$$
\begin{array}{r}
2x^2 - 9x - 18 \\
x+3 \overline{\big) 2x^3 - 3x^2 - 45x - 54} \\
\underline{2x^3 + 6x^2} \qquad\qquad\qquad \\
-9x^2 - 45x \qquad \\
\underline{-9x^2 - 27x} \qquad \\
-18x - 54 \\
\underline{-18x - 54} \\
0
\end{array}
$$

$f(x) = (x - 1)(x + 3)(2x^2 - 9x - 18)$
$= (x - 1)(x + 3)(2x + 3)(x - 6)$
$0 = (x - 1)(x + 3)(2x + 3)(x - 6)$
The solutions are $x = 1$, $x = -3$, $x = -\frac{3}{2}$ and $x = 6$.