Circles, Mixed Exercise 6

1  a  $QR$ is the diameter of the circle so the centre, $C$, is the midpoint of $QR$
Midpoint = \( \left( \frac{11+(-5)}{2}, \frac{12+0}{2} \right) = (3, 6) \)

$C(3, 6)$

b  Radius = \( \frac{1}{2} \) of diameter = \( \frac{1}{2} \) of $QR$ = \( \frac{1}{2} \) of $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$
\[= \frac{1}{2} \text{ of } \sqrt{(-5-11)^2 + (0-12)^2}\]
\[= \frac{1}{2} \text{ of } \sqrt{400}\]
\[= \frac{1}{2} \text{ of } 20 = 10 \text{ units}\]

c  Circle with centre $(3, 6)$ and radius 10:

\[(x-3)^2 + (y-6)^2 = 100\]

d  $P(13, 6)$ lies on the circle if $P$ satisfies the equation, so substitute $x = 13$ and $y = 6$ into the equation of the circle:

\[(13-3)^2 + (6-6)^2 = 100 + 0 = 100\]
Therefore, $P$ lies on the circle.

2  The distance between $(0, 0)$ and $(5, -2)$ is
\[\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2} = \sqrt{(5-0)^2 + (-2-0)^2} = \sqrt{25 + 4} = \sqrt{29}\]
The radius of the circle is $\sqrt{29}$.
As $\sqrt{29} < \sqrt{30}$, $(0,0)$ lies inside the circle.

3  a  $x^2 + 3x + y^2 + 6y = 3x - 2y - 7$
\[x^2 + y^2 + 8y = -7\]
Completing the square gives:
\[(x-0)^2 + (y+4)^2 - 16 = -7\]
\[(x-0)^2 + (y+4)^2 = 9\]
Centre of the circle is $(0, -4)$ and the radius is 3.

b  The circle intersects the $y$-axis at $x = 0$
\[(0-0)^2 + (y+4)^2 = 9\]
\[y^2 + 8y + 16 = 9\]
\[y^2 + 8y + 7 = 0\]
\[(y+1)(y+7) = 0\]
y = -1 or y = -7
$(0, -1)$ and $(0, -7)$
At the $x$-axis, $y = 0$

$x^2 + 0^2 + 8(0) = -7$

$x^2 = -7$

There are no real solutions, so the circle does not intersect the $x$-axis.

The centre of $(x-8) + (y-8)^2 = 117$ is $(8,8)$.

Substitute $(8, 8)$ into $(x+1)^2 + (y-3)^2 = 106$

$(8+1)^2 + (8-3)^2 = 9^2 + 5^2 = 81 + 25 = 106 \checkmark$

So $(8, 8)$ lies on the circle $(x+1)^2 + (y-3)^2 = 106$.

As $Q$ is the centre of the circle $(x+1)^2 + (y-3)^2 = 106$ and $P$ lies on this circle, the length $PQ$ must equal the radius.

So $PQ = \sqrt{106}$

Alternative method: Work out the distance between $P(8, 8)$ and $Q(-1, 3)$ using the distance formula.

Substitute $(-1, 0)$ into $x^2 + y^2 = 1$

$(-1)^2 + (0)^2 = 1 + 0 = 1 \checkmark$

So $(-1, 0)$ is on the circle.

Substitute $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ into $x^2 + y^2 = 1$

$\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1 \checkmark$

So $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is on the circle.

Substitute $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ into $x^2 + y^2 = 1$

$\left(\frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4} + \frac{3}{4} = 1 \checkmark$

So $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is on the circle.
5 b The distance between \((-1, 0)\) and \(\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\) is

\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{2} - (-1)\right)^2 + \left(\frac{\sqrt{3}}{2} - 0\right)^2}
\]

\[
= \sqrt{\left(\frac{1}{2} + 1\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}
\]

\[
= \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}
\]

\[
= \sqrt{\frac{9}{4} + \frac{3}{4}}
\]

\[
= \sqrt{\frac{12}{4}}
\]

\[
= \sqrt{3}
\]

The distance between \((-1, 0)\) and \(\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)\) is

\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{2} - (-1)\right)^2 + \left(-\frac{\sqrt{3}}{2} - 0\right)^2}
\]

\[
= \sqrt{\left(\frac{1}{2} + 1\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}
\]

\[
= \sqrt{\left(\frac{3}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2}\right)^2}
\]

\[
= \sqrt{\frac{9}{4} + \frac{3}{4}}
\]

\[
= \sqrt{\frac{12}{4}}
\]

\[
= \sqrt{3}
\]
5  b  

The distance between \( \left( \frac{1}{2}, \frac{\sqrt{3}}{2} \right) \) and \( \left( \frac{1}{2}, -\frac{\sqrt{3}}{2} \right) \) is

\[
\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{\left( \frac{1}{2} - \frac{1}{2} \right)^2 + \left( -\frac{\sqrt{3}}{2} - \frac{-\sqrt{3}}{2} \right)^2} = \sqrt{0^2 + (-\sqrt{3})^2} = \sqrt{0+3} = \sqrt{3}
\]

So \( AB, BC \) and \( AC \) all equal \( \sqrt{3} \).

\( \Delta ABC \) is equilateral.

6  a  

\((x-k)^2 + (y-3k)^2 = 13, \ (3, \ 0)\)

Substitute \( x = 3 \) and \( y = 0 \) into the equation of the circle.

\[
(3-k)^2 + (0-3k)^2 = 13
\]

\[
9 - 6k + k^2 + 9k^2 - 13 = 0
\]

\[
10k^2 - 6k - 4 = 0
\]

\[
5k^2 - 3k - 2 = 0
\]

\( (5k+2)(k-1) = 0 \)

\( k = -\frac{2}{5} \) or \( k = 1 \)

b  

As \( k > 0 \), \( k = 1 \)

Equation of the circle is \((x-1)^2 + (y-3)^2 = 13\)

7  

\( x^2 + px + y^2 + 4y = 20, \ y = 3x - 9 \)

Substitute \( y = 3x - 9 \) into the equation \( x^2 + px + y^2 + 4y = 20 \)

\[
x^2 + px + (3x - 9)^2 + 4(3x - 9) = 20
\]

\[
x^2 + px + 9x^2 - 54x + 81 + 12x - 36 - 20 = 0
\]

\[
10x^2 + (p - 42)x + 25 = 0
\]

There are no solutions, so using the discriminant \( b^2 - 4ac < 0 \):

\[
(p - 42)^2 - 4(10)(25) < 0
\]

\[
(p - 42)^2 < 1000
\]

\[
p - 42 < \pm\sqrt{1000}
\]

\[
p < 42 \pm\sqrt{1000}
\]

\[
p < 42 \pm 10\sqrt{10}
\]

\[
42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}
\]
8 Substitute \( x = 0 \) into \( y = 2x - 8 \)
\[
y = 2(0) - 8 = -8
\]

Substitute \( y = 0 \) into \( y = 2x - 8 \)
\[
0 = 2x - 8
2x = 8
x = 4
\]

The line meets the coordinate axes at \((0, -8)\) and \((4, 0)\).

The coordinates of the centre of the circle are at the midpoint:
\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{0 + 4}{2}, \frac{-8 + 0}{2} \right) = \left( \frac{4}{2}, \frac{-8}{2} \right) = (2, -4)
\]

The length of the diameter is
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4 - 0)^2 + (0 - (-8))^2} = \sqrt{4^2 + 8^2} = \sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}
\]

So the length of the radius is \( \frac{4\sqrt{5}}{2} = 2\sqrt{5} \).

The centre of the circle is \((2, -4)\) and the radius is \(2\sqrt{5}\).

The equation of the circle is
\[
(x - x_1)^2 + (y - y_1)^2 = r^2
\]
\[
(x - 2)^2 + (y - (-4))^2 = (2\sqrt{5})^2
\]
\[
(x - 2)^2 + (y + 4)^2 = 20
\]

9 a The radius is
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(8 - 4)^2 + (10 - 0)^2} = \sqrt{4^2 + 10^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}
\]

b
9  b  The centre is on the perpendicular bisector of \((4, 0)\) and \((a, 0)\). So
\[
\frac{4 + a}{2} = 8
\]
\[
4 + a = 16
\]
\[
a = 12
\]

10  Substitute \(y = 0\) into \((x - 5)^2 + y^2 = 36\)

\[
(x - 5)^2 = 36
\]
\[
x - 5 = \sqrt{36}
\]
\[
x - 5 = \pm 6
\]
So \(x = 11\) and \(x = -1\)

The coordinates of \(P\) and \(Q\) are \((-1, 0)\) and \((11, 0)\).

11  Substitute \(x = 0\) into \((x + 4)^2 + (y - 7)^2 = 121\)

\[
4^2 + (y - 7)^2 = 121
\]
\[
16 + (y - 7)^2 = 121
\]
\[
(y - 7)^2 = 105
\]
\[
y - 7 = \pm \sqrt{105}
\]
So \(y = 7 \pm \sqrt{105}\)

The values of \(m\) and \(n\) are \(7 + \sqrt{105}\) and \(7 - \sqrt{105}\).

12  a  \((x + 5)^2 + (y + 2)^2 = 125\), \(A(a, 0), B(0, b)\)

At \(A(a, 0)\): \((a + 5)^2 + (0 + 2)^2 = 125\)
\[
a^2 + 10a + 25 + 4 - 125 = 0
\]
\[
a^2 + 10a - 96 = 0
\]
\[
(a + 16)(a - 6) = 0
\]
As \(a > 0\), \(a = 6\)

At \(B(0, b)\): \((0 + 5)^2 + (b + 2)^2 = 125\)
\[
25 + b^2 + 4b + 4 - 125 = 0
\]
\[
b^2 + 4b - 96 = 0
\]
\[
(b + 12)(b - 8) = 0
\]
As \(b > 0\), \(b = 8\)

So \(a = 6\), \(b = 8\)
12 b  $A(6, 0), B(0, 8)$

gradient = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 0}{0 - 6} = -\frac{4}{3}$

$y$-intercept = 8

Equation of the line $AB$ is $y = -\frac{4}{3}x + 8$

c  Area of triangle $OAB = \frac{1}{2} \times 6 \times 8 = 24 \text{ units}^2$

13 a  By symmetry $p = 0$.  

![Diagram of circle and coordinates](image)

Using Pythagoras’ theorem

$q^2 + 7^2 = 25^2$
$q^2 + 49 = 625$
$q^2 = 576$
$q = \pm\sqrt{576}$
$q = \pm24$

As $q > 0$, $q = 24$.

b  The circle meets the $y$-axis at $q \pm r$; i.e.

at $24 + 25 = 49$
and $24 - 25 = -1$

So the coordinates are $(0, 49)$ and $(0, -1)$. 
14 The gradient of the line joining \((-3, -7)\) and \((5, 1)\) is 

\[
\frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-7)}{5 - (-3)} = \frac{1 + 7}{5 + 3} = \frac{8}{8} = 1
\]

So the gradient of the tangent is \(-\frac{1}{1} = -1\).

The equation of the tangent is

\[y - y_1 = m(x - x_1)\]

\[y - (-7) = -1(x - (-3))\]

\[y + 7 = -1(x + 3)\]

\[y + 7 = -x - 3\]

\[y = -x - 10\] or \[x + y + 10 = 0\]

15

Let the coordinates of \(C\) be \((p, q)\).

\((2, -1)\) is the mid-point of \((3, 7)\) and \((p, q)\)

So \[\frac{3 + p}{2} = 2\] and \[\frac{7 + q}{2} = -1\]

\[\frac{3 + p}{2} = 2\]

\[3 + p = 4\]

\[p = 1\]

\[\frac{7 + q}{2} = -1\]

\[7 + q = -2\]

\[q = -9\]
15 So the coordinates of C are (1, −9).

The length of $AB$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(-5-3)^2+(3-7)^2} = \sqrt{(-8)^2+(-4)^2} = \sqrt{64+16} = \sqrt{80}$$

The length of $BC$ is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(-5-1)^2+(3-(-9))^2} = \sqrt{(-6)^2+(12)^2} = \sqrt{36+144} = \sqrt{180}$$

The area of $\Delta ABC$ is

$$\frac{1}{2} \sqrt{180} \sqrt{80} = \frac{1}{2} \sqrt{14400} = \frac{1}{2} \sqrt{144 \times 100} = \frac{1}{2} \sqrt{144} \times \sqrt{100} = \frac{1}{2} \times 12 \times 10 = 60$$

16 $(x - 6)^2 + (y - 5)^2 = 17$
Centre of the circle is $(6, 5)$.

Equation of the line touching the circle is $y = mx + 12$

Substitute the equation of the line into the equation of the circle:

$$(x - 6)^2 + (mx + 7)^2 = 17$$

$x^2 - 12x + 36 + m^2x^2 + 14mx + 49 - 17 = 0$

$(1 + m^2)x^2 + (14m - 12)x + 68 = 0$

There is one solution so using the discriminant $b^2 - 4ac = 0$:

$(14m - 12)^2 - 4(1 + m^2)(68) = 0$

$196m^2 - 336m + 144 - 272m^2 - 272 = 0$

$76m^2 + 336m + 128 = 0$

$19m^2 + 84m + 32 = 0$

$(19m + 8)(m + 4) = 0$

$m = -\frac{8}{19}$ or $m = -4$

$y = -\frac{8}{19}x + 12$ and $y = -4x + 12$

17 a Gradient of $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{1-7}{5-3} = -3$

Midpoint of $AB = \left( \frac{3+5}{2}, \frac{7+1}{2} \right) = (4, 4)$

$M(4, 4)$

Line $l$ is perpendicular to $AB$, so gradient of line $l = \frac{1}{3}$

$y - y_1 = m(x - x_1)$

$y - 4 = \frac{1}{3}(x - 4)$
17 a  \( y = \frac{1}{3}x + \frac{8}{3} \)

b  \( C(-2, c) \)
\[
 y = \frac{1}{3}(-2) + \frac{8}{3} = 2
\]

\( C(-2, 2) \)

Radius of the circle = distance \( CA \)
\[
= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (2-7)^2} = \sqrt{50}
\]
Equation of the circle is \( (x + 2)^2 + (y - 2)^2 = 50 \)

c  Base of triangle = distance \( AB = \sqrt{(5-3)^2 + (1-7)^2} = \sqrt{40} \)

Height of triangle = distance \( CM = \sqrt{(4+2)^2 + (4-2)^2} = \sqrt{40} \)

Area of triangle \( ABC = \frac{1}{2} \times \sqrt{40} \times \sqrt{40} = 20 \) units\(^2\)

18 a  \((x - 3)^2 + (y + 3)^2 = 52\)

The equations of the lines \( l_1 \) and \( l_2 \) are \( y = \frac{3}{2}x + c \)

Diameter of the circle that touches \( l_1 \) and \( l_2 \) has gradient \( -\frac{2}{3} \) and passes through the centre of the circle \((3, -3)\)
\[
 y = -\frac{2}{3}x + d
\]

\[-3 = -\frac{2}{3}(3) + d
\]
\[d = -1
\]

\( y = -\frac{2}{3}x - 1 \) is the equation of the diameter that touches \( l_1 \) and \( l_2 \).

Solve the equation of the diameter and circle simultaneously:
\[
(x - 3)^2 + (-\frac{2}{3}x + 2)^2 = 52
\]
\[
x^2 - 6x + 9 + \frac{4}{9}x^2 - \frac{8}{3}x + 4 - 52 = 0
\]
\[
\frac{13}{9}x^2 - \frac{26}{3}x - 39 = 0
\]
\[
13x^2 - 78x - 351 = 0
\]
\[
x^2 - 6x - 27 = 0
\]
\[
(x - 9)(x + 3) = 0
\]
\[x = 9 \text{ or } x = -3\]
18 a  When $x = 9, y = \frac{-2}{3} (9) - 1 = -7$

When $x = -3, y = \frac{-2}{3} (-3) - 1 = 1$

$(9, -7)$ and $(-3, 1)$ are the coordinates where the diameter touches lines $l_1$ and $l_2$.
$P(9, -7)$ and $Q(-3, 1)$

b  The equations of the lines $l_1$ and $l_2$ are $y = \frac{3}{2} x + c$

$l_1$ touches the circle at $(-3, 1)$:

$1 = \frac{3}{2} (-3) + c, c = \frac{11}{2}, \text{ so } y = \frac{3}{2} x + \frac{11}{2}$

$l_2$ touches the circle at $(9, -7)$:

$-7 = \frac{3}{2} (9) + c, c = -\frac{41}{2}, \text{ so } y = \frac{3}{2} x - \frac{41}{2}$

19 a  $x^2 + 6x + y^2 - 2y = 7$

Equation of the lines are $y = mx + 6$

Substitute $y = mx + 6$ into the equation of the circle:

$x^2 + 6x + (mx + 6)^2 - 2(mx + 6) = 7$

$x^2 + 6x + m^2 x^2 + 12mx + 36 - 2mx - 12 - 7 = 0$

$(1 + m^2)x^2 + (6 + 10m)x + 17 = 0$

There is one solution so using the discriminant $b^2 - 4ac = 0$:

$(6 + 10m)^2 - 4(1 + m^2)(17) = 0$

$100m^2 + 120m + 36 - 68m^2 - 68 = 0$

$32m^2 + 120m - 32 = 0$

$4m^2 + 15m - 4 = 0$

$(4m - 1)(m + 4) = 0$

$m = \frac{1}{4} \text{ or } m = -4$

$y = \frac{1}{4} x + 6$ and $y = -4x + 6$

b  The gradient of $l_1 = \frac{1}{4}$ and the gradient of $l_2 = -4$, so the two lines are perpendicular,

Therefore, $APRQ$ is a square.

$x^2 + 6x + y^2 - 2y = 7$

Completing the square:

$(x + 3)^2 - 9 + (y - 1)^2 - 1 = 7$

$(x + 3)^2 + (y - 1)^2 = 17$

Radius $= \sqrt{17}$
19 b  Let point P have the coordinates \((x, y)\)
Using Pythagoras' theorem:
\(l_2: (0 - x)^2 + (6 - y)^2 = 17\)

Using the equation for \(l_2, y = \frac{1}{4}x + 6,\) \((0 - x)^2 + \left(6 - \left(\frac{1}{4}x + 6\right)\right)^2 = 17\)

\[x^2 + \frac{1}{16}x^2 = 17\]
\[\frac{17}{16}x^2 = 17\]
\[x^2 = 16\]
\[x = \pm 4\]

From the diagram we know that \(x\) is negative, so \(x = -4, y = \frac{1}{4}(-4) + 6 = 5\)

\(P(-4, 5)\)

Now let point \(Q\) have the coordinates \((x, y)\).
Using the equation for \(l_1, y = -4x + 6,\) \((0 - x)^2 + (6 - (-4x + 6))^2 = 17\)

\[x^2 + 16x^2 = 17\]
\[17x^2 = 17\]
\[x^2 = 1\]
\[x = \pm 1\]

From the diagram \(x\) is positive, so \(x = 1, y = -4(1) + 6 = 2\)

\(Q(1, 2)\)

c  Area of the square = radius\(^2\) = 17 units\(^2\)

20 a  Equation of the circle: \((x - 6)^2 + (y - 9)^2 = 50\)

Equation of \(l_1: y = -x + 21\)

Substitute the equation of the line into the equation of the circle:
\[(x - 6)^2 + (-x + 12)^2 = 50\]
\[x^2 - 12x + 36 + x^2 - 24x + 144 - 50 = 0\]
\[2x^2 - 36x + 130 = 0\]
\[x^2 - 18x + 65 = 0\]
\[(x - 13)(x - 5) = 0\]

\(x = 13\) or \(x = 5\)

When \(x = 13, y = -13 + 21 = 8\)

When \(x = 5, y = -5 + 21 = 16\)

\(P(5, 16)\) and \(Q(13, 8)\)
20 b The gradient of the line $AP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-16}{6-5} = -7$

So the gradient of the line perpendicular to $AP$, $l_2$, is $\frac{1}{7}$.

The equation of the perpendicular line is

$$y - y_1 = m(x - x_1)$$

$m = \frac{1}{7}$ and $(x_1, y_1) = P(5, 16)$

So $y - 16 = \frac{1}{7}(x - 5)$

$$y = \frac{1}{7}x + \frac{107}{7}$$

The gradient of the line $AQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-8}{6-13} = -\frac{1}{7}$

So the gradient of the line perpendicular to $AQ$, $l_3$, is 7.

The equation of the perpendicular line is

$$y - y_1 = m(x - x_1)$$

$m = 7$ and $(x_1, y_1) = Q(13, 8)$

So $y - 8 = 7(x - 13)$

$y = 7x - 83$

$l_2$: $y = \frac{1}{7}x + \frac{107}{7}$ and $l_3$: $y = 7x - 83$

d $l_2: y = \frac{1}{7}x + \frac{107}{7}$, $l_3: y = 7x - 83$ and $l_4: y = x + 3$

Solve these equations simultaneously one pair at a time:

$l_2$ and $l_3$: $\frac{1}{7}x + \frac{107}{7} = 7x - 83$

$x + 107 = 49x - 581$

$48x = 688$

$x = \frac{43}{3}$, so $y = 7\left(\frac{43}{3}\right) - 83 = \frac{52}{3}$
20 d  \( l_2 \) and \( l_3 \) intersect at \( \left( \frac{43}{3}, \frac{52}{3} \right) \).

\( l_3 \) and \( l_4 \): \( 7x - 83 = x + 3 \)
\[ 6x = 86 \]
\[ x = \frac{43}{3}, \text{ so } y = \frac{43}{3} + 3 = \frac{52}{3} \]

Therefore all three lines intersect at \( R \left( \frac{43}{3}, \frac{52}{3} \right) \).

e  Area of kite \( APRQ = \frac{1}{2} \times AR \times PQ \)

\[ PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(13 - 5)^2 + (8 - 16)^2} = \sqrt{128} = 8\sqrt{2} \]
\[ AR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left( \frac{43}{3} - 6 \right)^2 + \left( \frac{52}{3} - 9 \right)^2} = \sqrt{\frac{1250}{9}} = \frac{25\sqrt{2}}{3} \]

Area = \( \frac{1}{2} \times \frac{25\sqrt{2}}{3} \times 8\sqrt{2} = \frac{200}{3} \)

21 a  \( y = -3x + 12 \)

Substitute \( x = 0 \) into \( y = -3x + 12 \)
\[ y = -3(0) + 12 = 12 \]
So \( A \) is \((0, 12)\)

Substitute \( y = 0 \) into \( y = -3x + 12 \)
\[ 0 = -3x + 12 \]
\[ 3x = 12 \]
\[ x = 4 \]
So \( B \) is \((4, 0)\).

b  The mid-point of \( AB \) is
\[ \left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{0 + 4}{2}, \frac{12 + 0}{2} \right) = (2, 6) \]

e
21 c \( \angle AOB = 90^\circ \), so \( AB \) is a diameter of the circle. The centre of the circle is the mid-point of \( AB \), i.e. \((2, 6)\). The length of the diameter \( AB \) is
\[
\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4 - 0)^2 + (0 - 12)^2} = \sqrt{4^2 + (-12)^2} = \sqrt{16 + 144} = \sqrt{160}
\]
So the radius of the circle is \( \frac{\sqrt{160}}{2} \).
The equation of the circle is
\[
(x - 2)^2 + (y - 6)^2 = \left(\frac{\sqrt{160}}{2}\right)^2
\]
\[
(x - 2)^2 + (y - 6)^2 = \frac{160}{4}
\]
\[
(x - 2)^2 + (y - 6)^2 = 40
\]

22 a \( A(-3, -2), B(-6, 0) \) and \( C(1, q) \)

\[
AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-6 + 3)^2 + (0 + 2)^2} = \sqrt{13}
\]
Diameter \( BC = \sqrt{(1 + 6)^2 + (q - 0)^2} = \sqrt{49 + q^2} \)

\[
AC = \sqrt{(1 + 3)^2 + (q + 2)^2} = \sqrt{16 + q^2 + 4q + 4} = \sqrt{q^2 + 4q + 20}
\]
Using Pythagoras' theorem \( AC^2 + AB^2 = BC^2 \):
\[
(q^2 + 4q + 20) + 13 = 49 + q^2
\]
\[
4q - 16 = 0
\]
\[
q = 4
\]

b The centre of the circle is the midpoint of \( B(-6, 0) \) and \( C(1, 4) \)

Midpoint \( BC = \left(\frac{-6 + 1}{2}, \frac{0 + 4}{2}\right) = \left(\frac{-5}{2}, \frac{2}{2}\right) \)
The radius is half of \( BC = \frac{1}{2} \) of \( \sqrt{49 + q^2} = \frac{1}{2} \) of \( \sqrt{49 + 4^2} = \frac{1}{2} \) of \( \sqrt{65} = \frac{\sqrt{65}}{2} \)
Equation of the circle is
\[
\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = \left(\left(\frac{\sqrt{65}}{2}\right)^2
\right)
\]
\[
\left(x + \frac{5}{2}\right)^2 + (y - 2)^2 = \frac{65}{4}
\]
23 a  \( R(-4, 3), S(7, 4) \) and \( T(8, -7) \)

\[
RT = \sqrt{(8 + 4)^2 + (-7 - 3)^2} = \sqrt{244}
\]

\[
RS = \sqrt{(7 + 4)^2 + (4 - 3)^2} = \sqrt{122}
\]

\[
ST = \sqrt{(8 - 7)^2 + (-7 - 4)^2} = \sqrt{122}
\]

Using Pythagoras' theorem, \( ST^2 + RS^2 = 122 + 122 = 244 = RT^2 \), therefore, \( RT \) is the diameter of the circle.

b  The radius of the circle is

\[
\frac{1}{2} \times \text{diameter} = \frac{1}{2} \times \sqrt{244} = \frac{1}{2} \times \sqrt{4 \times 61} = \frac{1}{2} \times \sqrt{4} \times \sqrt{61} = \frac{1}{2} \times 2 \times \sqrt{61} = \sqrt{61}
\]

The centre of the circle is the mid-point of \( RT \):

\[
\left( \frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2} \right) = \left( \frac{-4 + 8}{2}, \frac{3 + (-7)}{2} \right) = \left( \frac{4}{2}, \frac{-4}{2} \right) = (2, -2)
\]

So the equation of the circle is

\[
(x - 2)^2 + (y + 2)^2 = \left(\sqrt{61}\right)^2 \text{ or } (x - 2)^2 + (y + 2)^2 = 61
\]

24  \( A(-4, 0), B(4, 8) \) and \( C(6, 0) \)

The gradient of the line \( AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 0}{4 + 4} = 1 \)

So the gradient of the line perpendicular to \( AB \), is \(-1\).

Midpoint of \( AB = \left( \frac{-4 + 4}{2}, \frac{0 + 8}{2} \right) = (0, 4) \)

The equation of the perpendicular line through the midpoint of \( AB \) is

\[
y - y_1 = m(x - x_1)
\]

\[
m = -1 \text{ and } (x_1, y_1) = (0, 4)
\]

So \( y - 4 = -(x - 0) \)

\[
y = -x + 4
\]

The gradient of the line \( BC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0 - 8}{6 - 4} = -4 \)

So the gradient of the line perpendicular to \( BC \), is \( \frac{1}{4} \).

Midpoint of \( BC = \left( \frac{4 + 6}{2}, \frac{8 + 0}{2} \right) = (5, 4) \)
24 The equation of the perpendicular line through the midpoint of \( BC \) is
\[
y - y_1 = m(x - x_1)
\]
\[
m = \frac{1}{4} \quad \text{and} \quad (x_1, y_1) = (5, 4)
\]
So \( y - 4 = \frac{1}{4}(x - 5) \)
\[
y = \frac{1}{4}x + \frac{11}{4}
\]
Solving these two equations simultaneously will give the centre of the circle:
\[
-x + 4 = \frac{1}{4}x + \frac{11}{4}
-4x + 16 = x + 11
5x = 5
x = 1, \text{ so } y = -1 + 4 = 3
\]
The centre of the circle is \((1, 3)\).

The radius is the distance from the centre of the circle \((1, 3)\) to a point on the circumference \(C(6, 0)\):

\[
\text{Radius} = \sqrt{(6 - 1)^2 + (0 - 3)^2} = \sqrt{34}
\]
The equation of the circle is \((x - 1)^2 + (y - 3)^2 = 34\)

25 a i \(A(-7, 7)\) and \(B(1, 9)\)

The gradient of the line \(AB\) is \( m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9 - 7}{1 + 7} = \frac{1}{4} \)

So the gradient of the line perpendicular to \(AB\), is \(-4\).

Midpoint of \(AB\) is \(\left(\frac{-7 + 1}{2}, \frac{7 + 9}{2}\right) = (-3, 8)\)

The equation of the perpendicular line is
\[
y - y_1 = m(x - x_1)
\]
\[
m = -4 \quad \text{and} \quad (x_1, y_1) = (-3, 8)
\]
So \( y - 8 = -4(x + 3) \)
\[
y = -4x - 4
\]

ii \(C(3, 1)\) and \(D(-7, 1)\)

The line \(CD\) is \(y = 1\)

Midpoint of \(CD\) is \(\left(\frac{3 - 7}{2}, \frac{1 + 1}{2}\right) = (-2, 1)\)

The equation of the perpendicular line is \(x = -2\)
25 b The two perpendicular bisectors cross at the centre of the circle.
Solve \( y = -4x - 4 \) and \( x = -2 \) simultaneously:
\[ y = -4(-2) - 4 = 4 \]
The centre of the circle = \((-2, 4)\).
The radius is the distance from the centre of the circle \((-2, 4)\) to a point on the circumference \(C(3, 1)\):
\[
\text{Radius} = \sqrt{(3+2)^2 + (1-4)^2} = \sqrt{34}
\]
The equation of the circle is \((x + 2)^2 + (y - 4)^2 = 34\).

Challenge

a Solve \((x - 5)^2 + (y - 3)^2 = 20\) and \((x - 10)^2 + (y - 8)^2 = 10\) simultaneously:
\[ x^2 - 10x + 25 + y^2 - 6y + 9 = 20 \quad \text{and} \quad x^2 - 20x + 100 + y^2 - 16y + 64 = 10 \]
\[ x^2 - 10x + y^2 - 6y + 14 = 0 \quad \text{and} \quad x^2 - 20x + y^2 - 16y + 154 = 0 \]
\[ x^2 - 10x + y^2 - 6y + 14 = x^2 - 20x + y^2 - 16y + 154 \]
\[ -10x + 10y = 140 \]
\[ x + y = 14 \]
\[ x = y = 14 \]

b Solve \((x - 5)^2 + (y - 3)^2 = 20\) and \(x + y = 14\) simultaneously:
\[ (9 - y)^2 + (y - 3)^2 = 20 \]
\[ 81 - 18y + y^2 + y^2 - 6y + 9 = 20 \]
\[ 2y^2 - 24y + 70 = 0 \]
\[ y^2 - 12y + 35 = 0 \]
\[ (y - 5)(y - 7) = 0 \]
\[ y = 5 \text{ or } y = 7 \]
When \(y = 5\), \(x = 14 - 5 = 9\)
When \(y = 7\), \(x = 14 - 7 = 7\)
\(P(7, 7)\) and \(Q(9, 5)\)

c Area of kite \(APBQ = \frac{1}{2} \times PQ \times AB\)
\[ PQ = \sqrt{(9 - 7)^2 + (5 - 7)^2} = \sqrt{8} \]
\(A(5, 3)\) and \(B(10, 8)\)
\[ AB = \sqrt{(10 - 5)^2 + (8 - 3)^2} = \sqrt{50} \]
\[ \text{Area} = \frac{1}{2} \times \sqrt{8} \times \sqrt{50} = \frac{1}{2} \times \sqrt{400} = 10 \text{ units}^2 \]