Circles, Mixed Exercise 6

1 a QR is the diameter of the circle so the centre, C, is the midpoint of QR

Midpoint $=\left(\dfrac{11+(-5)}{2}, \dfrac{12+0}{2}\right) = (3, 6)$

$C(3, 6)$

b Radius $= \dfrac{1}{2}$ of diameter $= \dfrac{1}{2}$ of $QR = \dfrac{1}{2}$ of $\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

$= \dfrac{1}{2}$ of $\sqrt{(-5-11)^2+(0-12)^2}$

$= \dfrac{1}{2}$ of $\sqrt{400}$

$= \dfrac{1}{2}$ of $20 = 10$ units

c Circle with centre (3, 6) and radius 10:
$(x-3)^2 + (y-6)^2 = 100$

d $P(13, 6)$ lies on the circle if P satisfies the equation, so substitute $x = 13$ and $y = 6$ into the equation of the circle:
$(13-3)^2 + (6-6)^2 = 100 + 0 = 100$
Therefore, P lies on the circle.

2 The distance between (0, 0) and (5, –2) is

$\sqrt{\left(x_2-x_1\right)^2+\left(y_2-y_1\right)^2} = \sqrt{\left(5-0\right)^2+\left(-2-0\right)^2} = \sqrt{5^2+\left(-2\right)^2} = \sqrt{25+4} = \sqrt{29}$

The radius of the circle is $\sqrt{30}$.

As $\sqrt{29} < \sqrt{30}$ $(0, 0)$ lies inside the circle.

3 a $x^2 + 3x + y^2 + 6y = 3x - 2y - 7$
$x^2 + y^2 + 8y = -7$
Completing the square gives:
$(x-0)^2 + (y+4)^2 - 16 = -7$
$(x-0)^2 + (y+4)^2 = 9$
Centre of the circle is $(0, -4)$ and the radius is 3.

b The circle intersects the y-axis at $x = 0$
$(0-0)^2 + (y+4)^2 = 9$
$y^2 + 8y + 16 = 9$
$y^2 + 8y + 7 = 0$
$(y+1)(y+7) = 0$
$y = -1$ or $y = -7$
$(0, -1)$ and $(0, -7)$

3 c At the x-axis, $y = 0$

$x^2 + 0^2 + 8(0) = -7$

$x^2 = -7$

There are no real solutions, so the circle does not intersect the x-axis.

4 a The centre of $(x-8)+(y-8)^2 = 117$ is $(8,8)$.

Substitute (8, 8) into $(x+1)^2+(y-3)^2 = 106$

$(8+1)^2+(8-3)^2 = 9^2+5^2 = 81+25 = 106$ ✓

So (8, 8) lies on the circle $(x+1)^2+(y-3)^2 = 106$.

b As Q is the centre of the circle $(x+1)^2+(y-3)^2 = 106$ and P lies on this circle, the length PQ must equal the radius.

So $PQ = \sqrt{106}$

Alternative method: Work out the distance between $P(8, 8)$ and $Q(-1, 3)$ using the distance formula.

5 a Substitute $(-1,0)$ into $x^2+y^2 = 1$

$(-1)^2+(0)^2 = 1+0 = 1$ ✓

So $(-1,0)$ is on the circle.

Substitute $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ into $x^2+y^2 = 1$

$\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}+\frac{3}{4} = 1$ ✓

So $\left(\frac{1}{2},\frac{\sqrt{3}}{2}\right)$ is on the circle.

Substitute $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ into $x^2+y^2 = 1$

$\left(\frac{1}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2 = \frac{1}{4}+\frac{3}{4} = 1$ ✓

So $\left(\frac{1}{2},-\frac{\sqrt{3}}{2}\right)$ is on the circle.

5 b The distance between $(-1, 0)$ and $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(\frac{1}{2}-(-1)\right)^2+\left(\frac{\sqrt{3}}{2}-0\right)^2}\\ &= \sqrt{\left(\frac{1}{2}+1\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\left(\frac{3}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\frac{9}{4}+\frac{3}{4}}\\ &= \sqrt{\frac{12}{4}}\\ &= \sqrt{3}\end{aligned}$$

The distance between $(-1, 0)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is

$$\begin{aligned}\sqrt{(x_2-x_1)^2+(y_2-y_1)^2} &= \sqrt{\left(\frac{1}{2}-(-1)\right)^2+\left(-\frac{\sqrt{3}}{2}-0\right)^2}\\ &= \sqrt{\left(\frac{1}{2}+1\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\left(\frac{3}{2}\right)^2+\left(-\frac{\sqrt{3}}{2}\right)^2}\\ &= \sqrt{\frac{9}{4}+\frac{3}{4}}\\ &= \sqrt{\frac{12}{4}}\\ &= \sqrt{3}\end{aligned}$$

5 b The distance between $\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ and $\left(\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{1}{2} - \frac{1}{2}\right)^2 + \left(-\frac{\sqrt{3}}{2} - \frac{\sqrt{3}}{2}\right)^2}$$
$$= \sqrt{0^2 + \left(-\sqrt{3}\right)^2}$$
$$= \sqrt{0 + 3}$$
$$= \sqrt{3}$$

So AB, BC and AC all equal $\sqrt{3}$.
ΔABC is equilateral.

6 a $(x - k)^2 + (y - 3k)^2 = 13$, $(3, 0)$
Substitute $x = 3$ and $y = 0$ into the equation of the circle.
$(3 - k)^2 + (0 - 3k)^2 = 13$
$9 - 6k + k^2 + 9k^2 - 13 = 0$
$10k^2 - 6k - 4 = 0$
$5k^2 - 3k - 2 = 0$
$(5k + 2)(k - 1) = 0$
$k = -\frac{2}{5}$ or $k = 1$

b As $k > 0$, $k = 1$
Equation of the circle is $(x - 1)^2 + (y - 3)^2 = 13$

7 $x^2 + px + y^2 + 4y = 20$, $y = 3x - 9$
Substitute $y = 3x - 9$ into the equation $x^2 + px + y^2 + 4y = 20$
$x^2 + px + (3x - 9)^2 + 4(3x - 9) = 20$
$x^2 + px + 9x^2 - 54x + 81 + 12x - 36 - 20 = 0$
$10x^2 + (p - 42)x + 25 = 0$
There are no solutions, so using the discriminant $b^2 - 4ac < 0$:
$(p - 42)^2 - 4(10)(25) < 0$
$(p - 42)^2 < 1000$
$p - 42 < \pm\sqrt{1000}$
$p < 42 \pm \sqrt{1000}$
$p < 42 \pm 10\sqrt{10}$
$42 - 10\sqrt{10} < p < 42 + 10\sqrt{10}$

8 Substitute $x = 0$ into $y = 2x - 8$
$y = 2(0) - 8$
$y = -8$

Substitute $y = 0$ into $y = 2x - 8$
$0 = 2x - 8$
$2x = 8$
$x = 4$

The line meets the coordinate axes at $(0, -8)$ and $(4, 0)$.
The coordinates of the centre of the circle are at the midpoint:

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0+4}{2}, \frac{-8+0}{2}\right) = \left(\frac{4}{2}, \frac{-8}{2}\right) = (2, -4)$$

The length of the diameter is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-0)^2 + (0-(-8))^2} = \sqrt{4^2 + 8^2} = \sqrt{80} = \sqrt{16 \times 5} = 4\sqrt{5}$$

So the length of the radius is $\dfrac{4\sqrt{5}}{2} = 2\sqrt{5}$.

The centre of the circle is $(2, -4)$ and the radius is $2\sqrt{5}$.

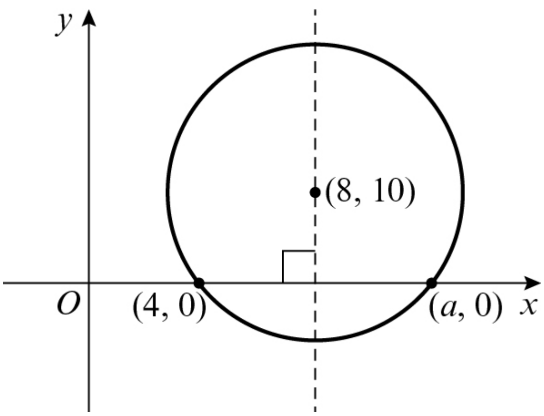
The equation of the circle is

$$(x - x_1)^2 + (y - y_1)^2 = r^2$$
$$(x-2)^2 + (y-(-4))^2 = (2\sqrt{5})^2$$
$$(x-2)^2 + (y+4)^2 = 20$$

9 a The radius is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(8-4)^2 + (10-0)^2} = \sqrt{4^2 + 10^2} = \sqrt{16 + 100} = \sqrt{116} = 2\sqrt{29}$$

b



9 b The centre is on the perpendicular bisector of $(4,0)$ and $(a,0.)$ So

$$\frac{4+a}{2}=8$$
$$4+a=16$$
$$a=12$$

10 Substitute $y = 0$ into $(x-5)^2+y^2=36$

$$(x-5)^2=36$$
$$x-5=\sqrt{36}$$
$$x-5=\pm 6$$
So $x-5=6 \Rightarrow x=11$
and $x-5=-6 \Rightarrow x=-1$

The coordinates of P and Q are $(-1,0)$ and $(11,0)$.

11 Substitute $x = 0$ into $(x+4)^2+(y-7)^2=121$

$$4^2+(y-7)^2=121$$
$$16+(y-7)^2=121$$
$$(y-7)^2=105$$
$$y-7=\pm\sqrt{105}$$
So $y=7\pm\sqrt{105}$

The values of m and n are $7+\sqrt{105}$ and $7-\sqrt{105}$.

12 a $(x + 5)^2 + (y + 2)^2 = 125$, $A(a, 0)$, $B(0, b)$

At $A(a, 0)$: $(a + 5)^2 + (0 + 2)^2 = 125$
$$a^2 + 10a + 25 + 4 - 125 = 0$$
$$a^2 + 10a - 96 = 0$$
$$(a + 16)(a - 6) = 0$$
As $a > 0$, $a = 6$

At $B(0, b)$: $(0 + 5)^2 + (b + 2)^2 = 125$
$$25 + b^2 + 4b + 4 - 125 = 0$$
$$b^2 + 4b - 96 = 0$$
$$(b + 12)(b - 8) = 0$$
As $b > 0$, $b = 8$
So $a = 6$, $b = 8$

12 b $A(6, 0)$, $B(0, 8)$

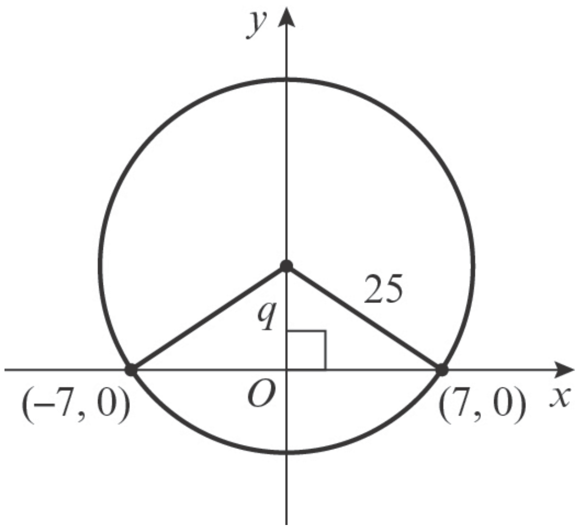
gradient = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 0}{0 - 6} = -\frac{4}{3}$

y-intercept = 8

Equation of the line AB is $y = -\frac{4}{3}x + 8$

c Area of triangle $OAB = \frac{1}{2} \times 6 \times 8 = 24$ units^2

13 a By symmetry $p = 0$.



Using Pythagoras' theorem

$$q^2 + 7^2 = 25^2$$
$$q^2 + 49 = 625$$
$$q^2 = 576$$
$$q = \pm\sqrt{576}$$
$$q = \pm 24$$

As $q > 0$, $q = 24$.

b The circle meets the y-axis at $q \pm r$; i.e.

at $24 + 25 = 49$

and $24 - 25 = -1$

So the coordinates are (0, 49) and (0, –1).

14 The gradient of the line joining $(-3,-7)$ and $(5,1)$ is

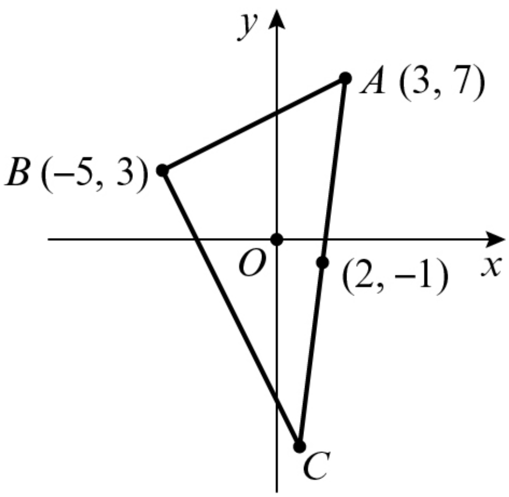
$$\frac{y_2-y_1}{x_2-x_1}=\frac{1-(-7)}{5-(-3)}=\frac{1+7}{5+3}=\frac{8}{8}=1$$

So the gradient of the tangent is $-\frac{1}{(1)}=-1$.

The equation of the tangent is

$$y-y_1=m(x-x_1)$$
$$y-(-7)=-1(x-(-3))$$
$$y+7=-1(x+3)$$
$$y+7=-x-3$$
$$y=-x-10 \text{ or } x+y+10=0$$

15



Let the coordinates of C be (p, q).
(2, –1) is the mid-point of (3, 7) and (p, q)

So $\frac{3+p}{2}=2$ and $\frac{7+q}{2}=-1$

$$\frac{3+p}{2}=2$$
$$3+p=4$$
$$p=1$$
$$\frac{7+q}{2}=-1$$
$$7+q=-2$$
$$q=-9$$

15 So the coordinates of C are (1, –9).

The length of *AB* is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(-5-3)^2+(3-7)^2}=\sqrt{(-8)^2+(-4)^2}=\sqrt{64+16}=\sqrt{80}$$

The length of *BC* is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(-5-1)^2+(3-(-9))^2}=\sqrt{(-6)^2+(12)^2}=\sqrt{36+144}=\sqrt{180}$$

The area of ΔABC is

$$\frac{1}{2}\sqrt{180}\sqrt{80}=\frac{1}{2}\sqrt{14400}=\frac{1}{2}\sqrt{144\times100}=\frac{1}{2}\sqrt{144}\times\sqrt{100}=\frac{1}{2}\times12\times10=60$$

16 $(x-6)^2+(y-5)^2=17$
Centre of the circle is (6, 5).

Equation of the line touching the circle is $y = mx + 12$

Substitute the equation of the line into the equation of the circle:

$$(x-6)^2+(mx+7)^2=17$$
$$x^2-12x+36+m^2x^2+14mx+49-17=0$$
$$(1+m^2)x^2+(14m-12)x+68=0$$

There is one solution so using the discriminant $b^2-4ac=0$:

$$(14m-12)^2-4(1+m^2)(68)=0$$
$$196m^2-336m+144-272m^2-272=0$$
$$76m^2+336m+128=0$$
$$19m^2+84m+32=0$$
$$(19m+8)(m+4)=0$$

$m=-\frac{8}{19}$ or $m=-4$

$y=-\frac{8}{19}x+12$ and $y=-4x+12$

17 a Gradient of *AB* = $\frac{y_2-y_1}{x_2-x_1}=\frac{1-7}{5-3}=-3$

Midpoint of *AB* = $\left(\frac{3+5}{2},\frac{7+1}{2}\right)$ = (4, 4)

$M(4, 4)$

Line l is perpendicular to *AB*, so gradient of line $l=\frac{1}{3}$

$y-y_1=m(x-x_1)$

$y-4=\frac{1}{3}(x-4)$

17 a $y = \frac{1}{3}x + \frac{8}{3}$

b $C(-2, c)$

$y = \frac{1}{3}(-2) + \frac{8}{3} = 2$

$C(-2, 2)$

Radius of the circle = distance CA

$= \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (2-7)^2} = \sqrt{50}$

Equation of the circle is $(x + 2)^2 + (y - 2)^2 = 50$

c Base of triangle = distance $AB = \sqrt{(5-3)^2 + (1-7)^2} = \sqrt{40}$

Height of triangle = distance $CM = \sqrt{(4+2)^2 + (4-2)^2} = \sqrt{40}$

Area of triangle $ABC = \frac{1}{2} \times \sqrt{40} \times \sqrt{40} = 20$ units^2

18 a $(x - 3)^2 + (y + 3)^2 = 52$

The equations of the lines l_1 and l_2 are $y = \frac{3}{2}x + c$

Diameter of the circle that touches l_1 and l_2 has gradient $-\frac{2}{3}$ and passes through the centre of the circle $(3, -3)$

$y = -\frac{2}{3}x + d$

$-3 = -\frac{2}{3}(3) + d$

$d = -1$

$y = -\frac{2}{3}x - 1$ is the equation of the diameter that touches l_1 and l_2.

Solve the equation of the diameter and circle simultaneously:

$$(x - 3)^2 + (-\frac{2}{3}x + 2)^2 = 52$$

$$x^2 - 6x + 9 + \frac{4}{9}x^2 - \frac{8}{3}x + 4 - 52 = 0$$

$$\frac{13}{9}x^2 - \frac{26}{3}x - 39 = 0$$

$$13x^2 - 78x - 351 = 0$$

$$x^2 - 6x - 27 = 0$$

$$(x - 9)(x + 3) = 0$$

$$x = 9 \text{ or } x = -3$$

18 a When $x = 9$, $y = -\frac{2}{3}(9) - 1 = -7$

When $x = -3$, $y = -\frac{2}{3}(-3) - 1 = 1$

$(9, -7)$ and $(-3, 1)$ are the coordinates where the diameter touches lines l_1 and l_2.
$P(9, -7)$ and $Q(-3, 1)$

b The equations of the lines l_1 and l_2 are $y = \frac{3}{2}x + c$

l_1 touches the circle at $(-3, 1)$:

$1 = \frac{3}{2}(-3) + c$, $c = \frac{11}{2}$, so $y = \frac{3}{2}x + \frac{11}{2}$

l_2 touches the circle at $(9, -7)$:

$-7 = \frac{3}{2}(9) + c$, $c = -\frac{41}{2}$, so $y = \frac{3}{2}x - \frac{41}{2}$

19 a $x^2 + 6x + y^2 - 2y = 7$
Equation of the lines are $y = mx + 6$

Substitute $y = mx + 6$ into the equation of the circle:

$$x^2 + 6x + (mx + 6)^2 - 2(mx + 6) = 7$$
$$x^2 + 6x + m^2x^2 + 12mx + 36 - 2mx - 12 - 7 = 0$$
$$(1 + m^2)x^2 + (6 + 10m)x + 17 = 0$$

There is one solution so using the discriminant $b^2 - 4ac = 0$:

$$(6 + 10m)^2 - 4(1 + m^2)(17) = 0$$
$$100m^2 + 120m + 36 - 68m^2 - 68 = 0$$
$$32m^2 + 120m - 32 = 0$$
$$4m^2 + 15m - 4 = 0$$
$$(4m - 1)(m + 4) = 0$$

$m = \frac{1}{4}$ or $m = -4$

$y = \frac{1}{4}x + 6$ and $y = -4x + 6$

b The gradient of $l_1 = \frac{1}{4}$ and the gradient of $l_2 = -4$, so the two lines are perpendicular,

Therefore, $APRQ$ is a square.
$x^2 + 6x + y^2 - 2y = 7$
Completing the square:

$$(x + 3)^2 - 9 + (y - 1)^2 - 1 = 7$$
$$(x + 3)^2 + (y - 1)^2 = 17$$

Radius = $\sqrt{17}$

19 b Let point P have the coordinates (x, y)
Using Pythagoras' theorem:
l_2: $(0 - x)^2 + (6 - y)^2 = 17$

Using the equation for l_2, $y = \frac{1}{4}x + 6$, $(0-x)^2 + \left(6 - \left(\frac{1}{4}x + 6\right)\right)^2 = 17$

$$x^2 + \frac{1}{16}x^2 = 17$$
$$\frac{17}{16}x^2 = 17$$
$$x^2 = 16$$
$$x = \pm 4$$

From the diagram we know that x is negative, so $x = -4$, $y = \frac{1}{4}(-4) + 6 = 5$

$P(-4, 5)$

Now let point Q have the coordinates (x, y).
Using the equation for l_1, $y = -4x + 6$, $(0 - x)^2 + (6 - (-4x + 6))^2 = 17$

$$x^2 + 16x^2 = 17$$
$$17x^2 = 17$$
$$x^2 = 1$$
$$x = \pm 1$$

From the diagram x is positive, so $x = 1$, $y = -4(1) + 6 = 2$
$Q(1, 2)$

c Area of the square = radius^2 = 17 units^2

20 a Equation of the circle: $(x - 6)^2 + (y - 9)^2 = 50$

Equation of l_1: $y = -x + 21$

Substitute the equation of the line into the equation of the circle:

$$(x - 6)^2 + (-x + 12)^2 = 50$$
$$x^2 - 12x + 36 + x^2 - 24x + 144 - 50 = 0$$
$$2x^2 - 36x + 130 = 0$$
$$x^2 - 18x + 65 = 0$$
$$(x - 13)(x - 5) = 0$$

$x = 13$ or $x = 5$
When $x = 13$, $y = -13 + 21 = 8$
When $x = 5$, $y = -5 + 21 = 16$
$P(5, 16)$ and $Q(13, 8)$

20 b The gradient of the line $AP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-16}{6-5} = -7$

So the gradient of the line perpendicular to AP, l_2, is $\frac{1}{7}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{7}$ and $(x_1, y_1) = P(5, 16)$

So $y - 16 = \frac{1}{7}(x - 5)$

$y = \frac{1}{7}x + \frac{107}{7}$

The gradient of the line $AQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{9-8}{6-13} = -\frac{1}{7}$

So the gradient of the line perpendicular to AQ, l_3, is 7.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 7$ and $(x_1, y_1) = Q(13, 8)$

So $y - 8 = 7(x - 13)$

$y = 7x - 83$

l_2: $y = \frac{1}{7}x + \frac{107}{7}$ and l_3: $y = 7x - 83$

c The gradient of the line $PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-16}{13-5} = -1$

So the gradient of the line perpendicular to PQ, l_4, is 1.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = 1$ and $(x_1, y_1) = A(6, 9)$

So $y - 9 = 1(x - 6)$

l_4: $y = x + 3$

d l_2: $y = \frac{1}{7}x + \frac{107}{7}$, l_3: $y = 7x - 83$ and l_4: $y = x + 3$

Solve these equations simultaneously one pair at a time:

l_2 and l_3: $\frac{1}{7}x + \frac{107}{7} = 7x - 83$

$x + 107 = 49x - 581$

$48x = 688$

$x = \frac{43}{3}$, so $y = 7\left(\frac{43}{3}\right) - 83 = \frac{52}{3}$

20 d l_2 and l_3 intersect at $\left(\frac{43}{3}, \frac{52}{3}\right)$.

l_3 and l_4: $7x - 83 = x + 3$

$6x = 86$

$x = \frac{43}{3}$, so $y = \frac{43}{3} + 3 = \frac{52}{3}$

Therefore all three lines intersect at $R\left(\frac{43}{3}, \frac{52}{3}\right)$

e Area of kite $APRQ = \frac{1}{2} \times AR \times PQ$

$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(13-5)^2 + (8-16)^2} = \sqrt{128} = 8\sqrt{2}$

$AR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{\left(\frac{43}{3} - 6\right)^2 + \left(\frac{52}{3} - 9\right)^2} = \sqrt{\left(\frac{25}{3}\right)^2 + \left(\frac{25}{3}\right)^2} = \sqrt{\frac{1250}{9}} = \frac{25\sqrt{2}}{3}$

Area $= \frac{1}{2} \times \frac{25\sqrt{2}}{3} \times 8\sqrt{2} = \frac{200}{3}$

21 a $y = -3x + 12$

Substitute $x = 0$ into $y = -3x + 12$

$y = -3(0) + 12 = 12$

So A is (0, 12)

Substitute $y = 0$ into $y = -3x + 12$

$0 = -3x + 12$

$3x = 12$

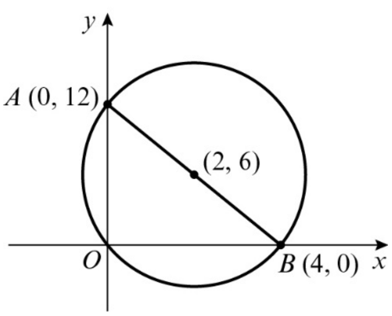
$x = 4$

So B is (4, 0).

b The mid-point of AB is

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0+4}{2}, \frac{12+0}{2}\right) = (2, 6)$

c



21 c $\angle AOB = 90°$, so AB is a diameter of the circle.
The centre of the circle is the mid-point of AB, i.e. (2, 6).
The length of the diameter AB is

$$\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}=\sqrt{(4-0)^2+(0-12)^2}=\sqrt{4^2+(-12)^2}=\sqrt{16+144}=\sqrt{160}$$

So the radius of the circle is $\dfrac{\sqrt{160}}{2}$.

The equation of the circle is

$$(x-2)^2+(y-6)^2=\left(\frac{\sqrt{160}}{2}\right)^2$$

$$(x-2)^2+(y-6)^2=\frac{160}{4}$$

$$(x-2)^2+(y-6)^2=40$$

22 a $A(-3, -2)$, $B(-6, 0)$ and $C(1, q)$

$$AB = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2} = \sqrt{(-6+3)^2+(0+2)^2} = \sqrt{13}$$

$$\text{Diameter} = BC = \sqrt{(1+6)^2+(q-0)^2} = \sqrt{49+q^2}$$

$$AC = \sqrt{(1+3)^2+(q+2)^2} = \sqrt{16+q^2+4q+4} = \sqrt{q^2+4q+20}$$

Using Pythagoras' theorem $AC^2 + AB^2 = BC^2$:

$$(q^2 + 4q + 20) + 13 = 49 + q^2$$
$$4q - 16 = 0$$
$$q = 4$$

b The centre of the circle is the midpoint of $B(-6, 0)$ and $C(1, 4)$

$$\text{Midpoint } BC = \left(\frac{-6+1}{2},\frac{0+4}{2}\right) = \left(-\frac{5}{2},2\right)$$

The radius is half of $BC = \frac{1}{2}$ of $\sqrt{49+q^2} = \frac{1}{2}$ of $\sqrt{49+4^2} = \frac{1}{2}$ of $\sqrt{65} = \frac{\sqrt{65}}{2}$

Equation of the circle is $\left(x+\frac{5}{2}\right)^2+(y-2)^2=\left(\frac{\sqrt{65}}{2}\right)^2$

$$\left(x+\frac{5}{2}\right)^2+(y-2)^2=\frac{65}{4}$$

23 a $R(-4, 3)$, $S(7, 4)$ and $T(8, -7)$

$RT = \sqrt{(8+4)^2 + (-7-3)^2} = \sqrt{244}$

$RS = \sqrt{(7+4)^2 + (4-3)^2} = \sqrt{122}$

$ST = \sqrt{(8-7)^2 + (-7-4)^2} = \sqrt{122}$

Using Pythagoras' theorem, $ST^2 + RS^2 = 122 + 122 = 244 = RT^2$, therefore, RT is the diameter of the circle.

b The radius of the circle is

$\frac{1}{2} \times \text{diameter} = \frac{1}{2}\sqrt{244} = \frac{1}{2}\sqrt{4 \times 61} = \frac{1}{2}\sqrt{4} \times \sqrt{61} = \frac{1}{2} \times 2\sqrt{61} = \sqrt{61}$

The centre of the circle is the mid-point of RT:

$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4+8}{2}, \frac{3+(-7)}{2}\right) = \left(\frac{4}{2}, \frac{-4}{2}\right) = (2, -2)$

So the equation of the circle is

$(x-2)^2 + (y+2)^2 = \left(\sqrt{61}\right)^2$ or $(x-2)^2 + (y+2)^2 = 61$

24 $A(-4, 0)$, $B(4, 8)$ and $C(6, 0)$

The gradient of the line $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8-0}{4+4} = 1$

So the gradient of the line perpendicular to AB, is -1.

Midpoint of $AB = \left(\frac{-4+4}{2}, \frac{0+8}{2}\right) = (0, 4)$

The equation of the perpendicular line through the midpoint of AB is

$y - y_1 = m(x - x_1)$

$m = -1$ and $(x_1, y_1) = (0, 4)$

So $y - 4 = -(x - 0)$

$y = -x + 4$

The gradient of the line $BC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{0-8}{6-4} = -4$

So the gradient of the line perpendicular to BC, is $\frac{1}{4}$.

Midpoint of $BC = \left(\frac{4+6}{2}, \frac{8+0}{2}\right) = (5, 4)$

24 The equation of the perpendicular line through the midpoint of *BC* is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{4}$ and $(x_1, y_1) = (5, 4)$

So $y - 4 = \frac{1}{4}(x - 5)$

$y = \frac{1}{4}x + \frac{11}{4}$

Solving these two equations simultaneously will give the centre of the circle:

$-x + 4 = \frac{1}{4}x + \frac{11}{4}$

$-4x + 16 = x + 11$

$5x = 5$

$x = 1$, so $y = -1 + 4 = 3$

The centre of the circle is (1, 3).

The radius is the distance from the centre of the circle (1, 3) to a point on the circumference *C*(6, 0):

Radius = $\sqrt{(6-1)^2 + (0-3)^2} = \sqrt{34}$

The equation of the circle is $(x - 1)^2 + (y - 3)^2 = 34$

25 a i *A*(−7, 7) and *B*(1, 9)

The gradient of the line *AB* = $\frac{y_2 - y_1}{x_2 - x_1} = \frac{9-7}{1+7} = \frac{1}{4}$

So the gradient of the line perpendicular to *AB*, is −4.

Midpoint of *AB* = $\left(\frac{-7+1}{2}, \frac{7+9}{2}\right) = (-3, 8)$

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -4$ and $(x_1, y_1) = (-3, 8)$

So $y - 8 = -4(x + 3)$

$y = -4x - 4$

ii *C*(3, 1) and *D*(−7, 1)

The line *CD* is $y = 1$

Midpoint of *CD* = $\left(\frac{3-7}{2}, \frac{1+1}{2}\right) = (-2, 1)$

The equation of the perpendicular line is $x = -2$

25 b The two perpendicular bisectors cross at the centre of the circle
Solve $y = -4x - 4$ and $x = -2$ simultaneously:
$y = -4(-2) - 4 = 4$
The centre of the circle = $(-2, 4)$
The radius is the distance from the centre of the circle $(-2, 4)$ to a point on the circumference $C(3, 1)$:
Radius = $\sqrt{(3+2)^2 + (1-4)^2} = \sqrt{34}$
The equation of the circle is $(x + 2)^2 + (y - 4)^2 = 34$

Challenge

a Solve $(x - 5)^2 + (y - 3)^2 = 20$ and $(x - 10)^2 + (y - 8)^2 = 10$ simultaneously:
$x^2 - 10x + 25 + y^2 - 6y + 9 = 20$ and $x^2 - 20x + 100 + y^2 - 16y + 64 = 10$
$x^2 - 10x + y^2 - 6y + 14 = 0$ and $x^2 - 20x + y^2 - 16y + 154 = 0$
$x^2 - 10x + y^2 - 6y + 14 = x^2 - 20x + y^2 - 16y + 154$
$-10x - 6y + 14 = -20x - 16y + 154$
$10x + 10y = 140$
$x + y = 14$
$x + y - 14 = 0$

b Solve $(x - 5)^2 + (y - 3)^2 = 20$ and $x + y = 14$ simultaneously:
$(9 - y)^2 + (y - 3)^2 = 20$
$81 - 18y + y^2 + y^2 - 6y + 9 = 20$
$2y^2 - 24y + 70 = 0$
$y^2 - 12y + 35 = 0$
$(y - 5)(y - 7) = 0$
$y = 5$ or $y = 7$

When $y = 5$, $x = 14 - 5 = 9$
When $y = 7$, $x = 14 - 7 = 7$

$P(7, 7)$ and $Q(9, 5)$

c Area of kite $APBQ = \frac{1}{2} \times PQ \times AB$
$PQ = \sqrt{(9-7)^2 + (5-7)^2} = \sqrt{8}$

$A(5, 3)$ and $B(10, 8)$

$AB = \sqrt{(10-5)^2 + (8-3)^2} = \sqrt{50}$

Area $= \frac{1}{2} \times \sqrt{8} \times \sqrt{50} = \frac{1}{2} \times \sqrt{400} = 10$ units^2