Circles 6F

1 a $U(-2, 8)$, $V(7, 7)$ and $W(-3, -1)$

$UV^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (7+2)^2 + (7-8)^2$

$= 82$

$VW^2 = (-3-7)^2 + (-1-7)^2$

$= 164$

$UW^2 = (-3+2)^2 + (-1-8)^2$

$= 82$

Use Pythagoras' theorem to show $UV^2 + UW^2 = VW^2$

$82 + 82 = 164 = VW^2$

Therefore, UVW is a right-angled triangle.

b UVW is a right-angled triangle, therefore VW is the diameter of the circle.

Centre of circle = Midpoint of VW

Midpoint = $\left(\frac{7+(-3)}{2}, \frac{7+(-1)}{2}\right) = (2, 3)$

c Radius of the circle is $\frac{1}{2}$ of $VW = \frac{\sqrt{164}}{2} = \sqrt{\frac{164}{4}} = \sqrt{41}$

$(x-2)^2 + (y-3)^2 = 41$

2 a $A(2, 6)$, $B(5, 7)$ and $C(8, -2)$

Use Pythagoras' theorem to show $AB^2 + BC^2 = AC^2$

$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (5-2)^2 + (7-6)^2 = 10$

$BC^2 = (8-5)^2 + (-2-7)^2 = 90$

$AC^2 = (8-2)^2 + (-2-6)^2 = 100$

Therefore, ABC is a right-angled triangle and AC is the diameter of the circle.

b Centre of circle = Midpoint of AC

Midpoint = $\left(\frac{2+8}{2}, \frac{6+(-2)}{2}\right) = (5, 2)$

Radius of the circle is $\frac{1}{2}$ of $AC = \frac{\sqrt{100}}{2} = 5$

$(x-5)^2 + (y-2)^2 = 25$

c Base of triangle = $AB = \sqrt{10}$ units

Height of triangle = $BC = \sqrt{90}$ units

Area of triangle $ABC = \frac{1}{2} \times \sqrt{10} \times \sqrt{90} = 15$ units^2

3 a i $A(-3, 19)$ and $B(9, 11)$

Midpoint = $\left(\frac{-3+9}{2}, \frac{19+11}{2}\right) = (3, 15)$

The gradient of the line segment $AB = \frac{y_2 - y_1}{x_2 - x_1} = \frac{11-19}{9-(-3)} = -\frac{2}{3}$

So the gradient of the line perpendicular to AB is $\frac{3}{2}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{3}{2}$ and $(x_1, y_1) = (3, 15)$

So $y - 15 = \frac{3}{2}(x - 3)$

$y = \frac{3}{2}x + \frac{21}{2}$

ii $A(-3, 19)$ and $C(-15, 1)$

Midpoint = $\left(\frac{-3-15}{2}, \frac{19+1}{2}\right) = (-9, 10)$

The gradient of the line segment $AC = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-19}{-15+3} = \frac{3}{2}$

So the gradient of the line perpendicular to AC is $-\frac{2}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -\frac{2}{3}$ and $(x_1, y_1) = (-9, 10)$

So $y - 10 = -\frac{2}{3}(x + 9)$

$y = -\frac{2}{3}x + 4$

b Solve $y = \frac{3}{2}x + \frac{21}{2}$ and $y = -\frac{2}{3}x + 4$ simultaneously

$\frac{3}{2}x + \frac{21}{2} = -\frac{2}{3}x + 4$

$9x + 63 = -4x + 24$

$13x = -39$

$x = -3$, $y = -\frac{2}{3}(-3) + 4 = 6$

So, the coordinates of the centre of the circle are $(-3, 6)$

3 c Radius = distance from (−3, 6) to (9, 11)

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(9+3)^2 + (11-6)^2} = \sqrt{12^2 + 5^2} = 13$

$(x + 3)^2 + (y - 6)^2 = 169$

4 a i $P(-11, 8)$ and $Q(-6, -7)$

Midpoint = $\left(\frac{-11-6}{2}, \frac{8-7}{2}\right) = \left(-\frac{17}{2}, \frac{1}{2}\right)$

The gradient of the line segment $PQ = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-7-8}{-6+11} = -3$

So the gradient of the line perpendicular to PQ is $\frac{1}{3}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{1}{3}$ and $(x_1, y_1) = \left(-\frac{17}{2}, \frac{1}{2}\right)$

So $y - \frac{1}{2} = \frac{1}{3}\left(x + \frac{17}{2}\right)$

$y = \frac{1}{3}x + \frac{10}{3}$

ii $Q(-6, -7)$ and $R(4, -7)$

QR is the line $y = -7$.

Midpoint = $\left(\frac{-6+4}{2}, \frac{-7-7}{2}\right) = (-1, -7)$

The equation of the perpendicular line is $x = -1$.

b Solve $y = \frac{1}{3}x + \frac{10}{3}$ and $x = -1$ simultaneously to find the centre of the circle:

$\frac{1}{3}(-1) + \frac{10}{3} = y$

$y = 3$

The centre of the circle is (−1, 3)

Radius = distance from (−1, 3) to (4, −7)

$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4+1)^2 + (-7-3)^2} = \sqrt{125}$

$(x + 1)^2 + (y - 3)^2 = 125$

5 $R(-2, 1)$ and $S(4, 3)$

Midpoint = $\left(\frac{-2+4}{2}, \frac{1+3}{2}\right)$ = (1, 2)

The gradient of the line segment $RS = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3-1}{4+2} = \frac{1}{3}$

So the gradient of the line perpendicular to RS is −3.
The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = -3$ and $(x_1, y_1) = (1, 2)$

So $y - 2 = -3(x - 1)$

$y = -3x + 5$

$S(4, 3)$ and $T(10, -5)$

Midpoint = $\left(\frac{4+10}{2}, \frac{3-5}{2}\right) = (7, -1)$

The gradient of the line segment $ST = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-5-3}{10-4} = -\frac{4}{3}$

So the gradient of the line perpendicular to ST is $\frac{3}{4}$.

The equation of the perpendicular line is

$y - y_1 = m(x - x_1)$

$m = \frac{3}{4}$ and $(x_1, y_1) = (7, -1)$

So $y + 1 = \frac{3}{4}(x - 7)$

$y = \frac{3}{4}x - \frac{25}{4}$

Solve $y = -3x + 5$ and $y = \frac{3}{4}x - \frac{25}{4}$ simultaneously

$-3x + 5 = \frac{3}{4}x - \frac{25}{4}$

$-12x + 20 = 3x - 25$

$15x = 45$

$x = 3$, $y = -3(3) + 5 = -4$

So the centre of the circle is (3, −4)

Radius = distance from centre (3, −4) to (−2, 1)

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-2-3)^2 + (1+4)^2} = \sqrt{50}$$

The equation of the circle is $(x - 3)^2 + (y + 4)^2 = 50$

6 a $A(3, 15)$, $B(-13, 3)$ and $C(-7, -5)$
Using Pythagoras' theorem $AB^2 + BC^2 = AC^2$
$AB^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-13 - 3)^2 + (3 - 15)^2 = 256 + 144 = 400$
$BC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-7 + 13)^2 + (-5 - 3)^2 = 36 + 64 = 100$
$AC^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 = (-7 - 3)^2 + (-5 - 15)^2 = 100 + 400 = 500$
Therefore ABC is a right-angled triangle.

b Centre of circle = midpoint of $AC = \left(\frac{3-7}{2}, \frac{15-5}{2}\right) = (-2, 5)$

Radius = $\frac{1}{2}$ of $AC = \frac{1}{2}$ of $\sqrt{500} = \frac{10\sqrt{5}}{2} = 5\sqrt{5}$
Equation of circle: $(x + 2)^2 + (y - 5)^2 = (5\sqrt{5})^2$ or $(x + 2)^2 + (y - 5)^2 = 125$

c We know that A, B and C all lie on the circumference of the circle.

$D(8, 0)$, substitute $x = 8$ and $y = 0$ into the equation of the circle:

$(8 + 2)^2 + (0 - 5)^2 = 100 + 25 = 125$

Therefore, $D(8, 0)$ lies on the circumference of the circle $(x + 2)^2 + (y - 5)^2 = 125$

7 a $A(-1, 9)$, $B(6, 10)$, $C(7, 3)$, $D(0, 2)$

The length of AB is
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(6-(-1))^2 + (10-9)^2} = \sqrt{7^2 + 1^2} = \sqrt{49+1} = \sqrt{50}$

The length of BC is
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(7-6)^2 + (3-10)^2} = \sqrt{1^2 + (-7)^2} = \sqrt{1+49} = \sqrt{50}$

The length of CD is
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(0-7)^2 + (2-3)^2} = \sqrt{(-7)^2 + (-1)^2} = \sqrt{49+1} = \sqrt{50}$

The length of DA is
$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-1-0)^2 + (9-2)^2} = \sqrt{(-1)^2 + 7^2} = \sqrt{1+49} = \sqrt{50}$

The sides of the quadrilateral are equal.

The gradient of AB is
$\frac{y_2 - y_1}{x_2 - x_1} = \frac{10-9}{6-(-1)} = \frac{1}{7}$

7 a The gradient of BC is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-10}{7-6} = \frac{-7}{1} = -7$$

The product of the gradients $= \left(\frac{1}{7} \times -7\right) = -1.$

So the line AB is perpendicular to BC.
So the quadrilateral $ABCD$ is a square.

b The area $= \sqrt{50} \times \sqrt{50} = 50$

c The mid-point of AC is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-1+7}{2}, \frac{9+3}{2}\right) = \left(\frac{6}{2}, \frac{12}{2}\right) = (3, 6)$$

So the centre of the circle is (3, 6).

8 a $D(-12, -3)$, $E(-10, b)$, $F(2, -5)$
Using Pythagoras' theorem $DE^2 + EF^2 = DF^2$

$DE^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$= (-10 + 12)^2 + (b + 3)^2$
$= 4 + b^2 + 6b + 9$
$= b^2 + 6b + 13$

$EF^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$= (2 + 10)^2 + (-5 - b)^2$
$= 144 + b^2 + 10b + 25$
$= b^2 + 10b + 169$

$DF^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$
$= (2 + 12)^2 + (-5 + 3)^2$
$= 196 + 4$
$= 200$

$b^2 + 6b + 13 + b^2 + 10b + 169 = 200$
$2b^2 + 16b - 18 = 0$
$b^2 + 8b - 9 = 0$
$(b + 9)(b - 1) = 0$
$b = -9$ or $b = 1$
As $b > 0$, $b = 1$.

b Centre of circle = midpoint of DF = $\left(\frac{-12+2}{2}, \frac{-3-5}{2}\right)$ = $(-5, -4)$

Distance of radius = $\frac{1}{2}$ of DF = $\frac{1}{2}$ of $\sqrt{200}$ = $\frac{10\sqrt{2}}{2}$ = $5\sqrt{2}$

Equation of circle: $(x + 5)^2 + (y + 4)^2 = (5\sqrt{2})^2 = 50$

9 a $x^2 + 2x + y^2 - 24y - 24 = 0$

Completing the square gives:

$(x + 1)^2 - 1 + (y - 12)^2 - 144 - 24 = 0$

$(x + 1)^2 + (y - 12)^2 = 169$

Centre of the circle is $(-1, 12)$ and the radius of the circle is 13.

b If AB is the diameter of the circle then the midpoint of AB is the centre of the circle.

Midpoint of $AB = \left(\frac{-13+11}{2}, \frac{17+7}{2}\right) = (-1, 12)$

Therefore, AB is the diameter of the circle.

c The point C lies on the x-axis, so $y = 0$.

Substitute $y = 0$ into the equation of the circle.

$(x + 1)^2 + (0 - 12)^2 = 169$

$x^2 + 2x + 1 + 144 = 169$

$x^2 + 2x - 24 = 0$

$(x + 6)(x - 4) = 0$

$x = -6, x = 4$

As x is negative, $x = -6$

The coordinates of C are $(-6, 0)$