Circles 6E

1 a $(x + 1)^2 + (y + 6)^2 = r^2$, (2, 3)
Substitute $x = 2$ and $y = 3$ into the equation $(x + 1)^2 + (y + 6)^2 = r^2$

$$(2+1)^2 + (3+6)^2 = r^2$$
$$9 + 81 = r^2$$
$$r = \sqrt{90}$$
$$= 3\sqrt{10}$$

b The line has equation $x + 3y - 11 = 0$
$3y = -x + 11$

$$y = -\frac{1}{3}x + \frac{11}{3}$$

The gradient of the line is $-\frac{1}{3}$

The gradient of the radius from the centre of the circle (−1, −6) to (2, 3) is:

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{3-(-6)}{2-(-1)} = \frac{9}{3} = 3$$

As $-\frac{1}{3} \times 3 = -1$, the line and the radius to the point (2, 3) are perpendicular.

2 a The radius of the circle is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(4-1)^2 + (6-(-2))^2} = \sqrt{3^2 + 8^2} = \sqrt{9+64} = \sqrt{73}$$

The equation of the circle is

$$(x-4)^2 + (y-6)^2 = \left(\sqrt{73}\right)^2$$

or $(x-4)^2 + (y-6)^2 = 73$

b The gradient of the line joining (1, –2) and (4, 6) is

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{6-(-2)}{4-1} = \frac{6+2}{3} = \frac{8}{3}$$

The gradient of the tangent is $\frac{-1}{\left(\frac{8}{3}\right)} = -\frac{3}{8}$

The equation of the tangent to the circle at (1, –2) is

$$y - y_1 = m(x - x_1)$$
$$y - (-2) = -\frac{3}{8}(x - 1)$$

2 b

$$y+2=-\frac{3}{8}(x-1)$$
$$8y+16=-3(x-1)$$
$$8y+16=-3x+3$$
$$3x+8y+16=3$$
$$3x+8y+13=0$$

3 a $A(-1, -9)$ and $B(7, -5)$

Midpoint = $\left(\frac{-1+7}{2}, \frac{-9+(-5)}{2}\right)$ = (3, −7)

The gradient of the line segment $AB = \frac{y_2-y_1}{x_2-x_1} = \frac{-5-(-9)}{7-(-1)} = \frac{1}{2}$

So the gradient of a line perpendicular to AB is −2.
The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

$m = -2$ and $(x_1, y_1) = (3, -7)$

So $y-(-7) = -2(x-3)$

$y = -2x - 1$

b Centre of the circle is (1, −3)
Substitute $x = 1$ into the equation $y = -2x - 1$
$y = -2(1) - 1 = -3$
Therefore, the perpendicular bisector of AB, $y = -2x - 1$, passes through the centre of the circle (1, −3)

4 a $P(3, 1)$ and $Q(5, -3)$

Midpoint = $\left(\frac{3+5}{2}, \frac{1+(-3)}{2}\right)$ = (4, −1)

The gradient of the line segment $PQ = \frac{y_2-y_1}{x_2-x_1} = \frac{-3-1}{5-3} = -2$

So the gradient of the line perpendicular to PQ is $\frac{1}{2}$.

The equation of the perpendicular line is

$y-y_1=m(x-x_1)$

m = $\frac{1}{2}$ and (x_1, y_1) = (4, −1)

So $y-(-1) = \frac{1}{2}(x-4)$

$y = \frac{1}{2}x - 3$

4 b Complete the square for $x^2 - 4x + y^2 + 4y = 2$

$(x - 2)^2 - 4 + (y + 2)^2 - 4 = 2$

$(x - 2)^2 + (y + 2)^2 = 10$

Centre of the circle is $(2, -2)$

Substitute $x = 2$ into the equation $y = \frac{1}{2}x - 3$

$y = \frac{1}{2}(2) - 3 = -2$

Therefore, the perpendicular bisector of PQ, $y = \frac{1}{2}x - 3$, passes through the centre of the circle $(2, -2)$

5 a Substitute $x = -7$ and $y = -6$ into $x^2 + 18x + y^2 - 2y + 29$

$$x^2 + 18x + y^2 - 2y + 29 = (-7)^2 + 18(-7) + (-6)^2 - 2(-6) + 29$$
$$= 49 - 126 + 36 + 12 + 29$$
$$= 0$$

The point P satisfies the equation, so P lies on C.

b Completing the square gives

$(x + 9)^2 + (y - 1)^2 = 53$

The centre of the circle, A, is $(-9, 1)$.

The gradient of $CP = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-6-1}{-7-(-9)} = \frac{-7}{2}$

Gradient of the tangent is $\frac{2}{7}$

$$y - y_1 = m(x - x_1)$$
$$y - (-6) = \frac{2}{7}(x - (-7))$$
$$y = \frac{2}{7}x - 4$$

c The tangent intersects the y-axis at $x = 0$

$y = \frac{2}{7}(0) - 4 = -4$

$R(0, -4)$

d Height of triangle = radius of circle = $\sqrt{53}$

Base of triangle = distance PR

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(-7-0)^2 + (-6-(-4))^2} = \sqrt{53}$$

Area $APR = \frac{1}{2} \times \sqrt{53} \times \sqrt{53} = 26.5$ units^2

6 a The centre of the circle $(x+4)^2+(y-1)^2=242$ is $(-4,1)$.

The gradient of the line joining (–4, 1) and (7, –10) is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{-10-1}{7-(-4)}=\frac{-11}{7+4}=-\frac{11}{11}=-1$$

The gradient of the tangent is $\frac{-1}{(-1)}=1$.

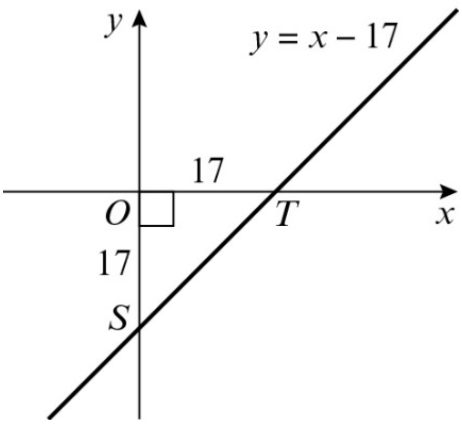
The equation of the tangent is

$$y-y_1=m(x-x_1)$$
$$y-(-10)=1(x-7)$$
$$y+10=x-7$$
$$y=x-17$$

Substitute $x = 0$ into $y = x - 17$
$y = 0 - 17$
$y = - 17$
So the coordinates of S are (0, –17)

Substitute $y = 0$ into $y = x - 17$
$0 = x - 17$
$x = 17$
So the coordinates of T are (17, 0).

b



The area of ΔOST is $\frac{1}{2}\times 17\times 17=144.5$

7 $(x + 5)^2 + (y + 3)^2 = 80$
Gradient of tangent = 2, so $y = 2x + c$

Diameter of the circle that touches l_1 and l_2 has gradient $-\frac{1}{2}$ and passes through the centre of the circle $(-5, -3)$

$y = -\frac{1}{2}x + d$

$-3 = -\frac{1}{2}(-5) + d$

$d = -\frac{11}{2}$

$y = -\frac{1}{2}x - \frac{11}{2}$ is the equation of the diameter that touches l_1 and l_2.

Solve the equation of the diameter and circle simultaneously:

$$(x + 5)^2 + (-\frac{1}{2}x - \frac{5}{2})^2 = 80$$

$$x^2 + 10x + 25 + \frac{1}{4}x^2 + \frac{5}{2}x + \frac{25}{4} - 80 = 0$$

$$4x^2 + 40x + 100 + x^2 + 10x + 25 - 320 = 0$$

$$5x^2 + 50x - 195 = 0$$

$$x^2 + 10x - 39 = 0$$

$$(x + 13)(x - 3) = 0$$

$x = -13$ or $x = 3$

When $x = -13$, $y = -\frac{1}{2}(-13) - \frac{11}{2} = 1$

When $x = 3$, $y = -\frac{1}{2}(3) - \frac{11}{2} = -7$

$(-13, 1)$ and $(3, -7)$ are the coordinates where the diameter touches lines l_1 and l_2.

Substitute these coordinates into the equation $y = 2x + c$

When $x = -13$, $y = 1$, $1 = 2(-13) + c$, $c = 27$, $y = 2x + 27$
When $x = 3$, $y = -7$, $-7 = 2(3) + c$, $c = -13$, $y = 2x - 13$
l_1: $y = 2x + 27$
l_2: $y = 2x - 13$

8 a $(x - 3)^3 + (y - p)^2 = 5$ and $2x + y - 5 = 0$
So $y = -2x + 5$

Solve the equations simultaneously:

$$(x - 3)^3 + (-2x + 5 - p)^2 = 5$$

$$x^2 - 6x + 9 + 4x^2 - 20x + 4px + 25 - 10p + p^2 - 5 = 0$$

$$5x^2 - 26x + 4px + 29 - 10p + p^2 = 0$$

8 a Using the discriminant for one solution:

$$b^2 - 4ac = 0$$
$$(-26 + 4p)^2 - 4(5)(29 - 10p + p^2) = 0$$
$$16p^2 - 208p + 676 - 20p^2 + 200p - 580 = 0$$
$$-4p^2 - 8p + 96 = 0$$
$$p^2 + 2p - 24 = 0$$
$$(p - 4)(p + 6) = 0$$

$p = 4$ or $p = -6$

b When $p = 4$, $(x - 3)^3 + (y - 4)^2 = 5$
When $p = - 6$, $(x - 3)^3 + (y + 6)^2 = 5$
(3, 4) and (3, −6)

9 a The centre of the circle, Q, is (11, −5)
To find the radius of the circle:

$$PQ = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(5 - 11)^2 + (3 - (-5))^2}$$
$$= \sqrt{(-6)^2 + 8^2}$$
$$= 10$$

$(x - 11)^2 + (y + 5)^2 = 100$

b The gradient of $PQ = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{3 - (-5)}{5 - 11} = \dfrac{4}{-3}$

Gradient of the tangent is $\dfrac{3}{4}$

$$y - y_1 = m(x - x_1)$$
$$y - 3 = \frac{3}{4}(x - 5)$$
$$y = \frac{3}{4}x - \frac{3}{4}$$

c Midpoint of $PQ = \left(\dfrac{11+5}{2}, \dfrac{-5+3}{2}\right) = (8, -1)$

Gradient of l_2 is $\dfrac{3}{4}$ as the line is parallel to l_1.

$$y - y_1 = m(x - x_1)$$
$$y - (-1) = \frac{3}{4}(x - 8)$$
$$y = \frac{3}{4}x - 7$$

9 c Solve $y = \frac{3}{4}x - 7$ and $(x-11)^2 + (y+5)^2 = 100$ simultaneously

$$(x-11)^2 + \left(\frac{3}{4}x - 2\right)^2 = 100$$

$$x^2 - 22x + 121 + \frac{9}{16}x^2 - 3x + 4 - 100 = 0$$
$$25x^2 - 400x + 400 = 0$$
$$x^2 - 16x + 16 = 0$$

Using the formula, $x = \frac{16 \pm \sqrt{192}}{2} = 8 \pm 4\sqrt{3}$

When $x = 8 + 4\sqrt{3}$, $y = \frac{3}{4}(8 + 4\sqrt{3}) - 7 = -1 + 3\sqrt{3}$

When $x = 8 - 4\sqrt{3}$, $y = \frac{3}{4}(8 - 4\sqrt{3}) - 7 = -1 - 3\sqrt{3}$

$A(8 - 4\sqrt{3},\ -1 - 3\sqrt{3})$ and $B(8 + 4\sqrt{3},\ -1 + 3\sqrt{3})$

d $AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

$= \sqrt{(8 + 4\sqrt{3} - (8 - 4\sqrt{3}))^2 + (-1 + 3\sqrt{3} - (-1 - 3\sqrt{3}))^2}$

$= \sqrt{(8\sqrt{3})^2 + (6\sqrt{3})^2}$

$= \sqrt{192 + 108}$

$= \sqrt{300}$

$= 10\sqrt{3}$

10 a $M = \left(\frac{2+10}{2}, \frac{3+1}{2}\right) = (6, 2)$

Gradient $RS = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-3}{10-2} = -\frac{1}{4}$

Gradient of l is 4

$y - y_1 = m(x - x_1)$

$y - 2 = 4(x - 6)$

$y = 4x - 22$ is the equation of l

b Using $y = 4x - 22$ when $x = a$, $y = -2$:
$-2 = 4a - 22$, $a = 5$

c radius = distance $CR = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2-5)^2 + (3-(-2))^2} = \sqrt{34}$

Centre of circle is $(5, -2)$

Equation of circle is $(x-5)^2 + (y+2)^2 = 34$

10 d Solve $y = 4x - 22$ and $(x - 5)^2 + (y + 2)^2 = 34$ simultaneously

$$(x - 5)^2 + (4x - 20)^2 = 34$$
$$x^2 - 10x + 25 + 16x^2 - 160x + 400 - 34 = 0$$
$$17x^2 - 170x + 391 = 0$$
$$x^2 - 10x + 23 = 0$$

Using the formula, $x = \dfrac{10 \pm \sqrt{8}}{2} = 5 \pm \sqrt{2}$

When $x = 5 + \sqrt{2}$, $y = 4(5 + \sqrt{2}) - 22 = -2 + 4\sqrt{2}$

When $x = 5 - \sqrt{2}$, $y = 4(5 - \sqrt{2}) - 22 = -2 - 4\sqrt{2}$

$A(5 + \sqrt{2}, -2 + 4\sqrt{2})$ and $B(5 - \sqrt{2}, -2 - 4\sqrt{2})$

11 a $x^2 - 4x + y^2 - 6y = 7$

$$(x - 2)^2 - 4 + (y - 3)^2 - 9 = 7$$
$$(x - 2)^2 + (y - 3)^2 = 20$$

Substitute $x = 3y - 17$ into $(x - 2)^2 + (y - 3)^2 = 20$

$$(3y - 19)^2 + (y - 3)^2 = 20$$
$$9y^2 - 114y + 361 + y^2 - 6y + 9 - 20 = 0$$
$$10y^2 - 120y + 350 = 0$$
$$y^2 - 12y + 35 = 0$$
$$(y - 7)(y - 5) = 0$$

$y = 7$ or 5

when $y = 7$, $x = 3(7) - 17 = 4$

when $y = 5$, $x = 3(5) - 17 = -2$

$P(-2, 5)$ and $Q(4, 7)$

b Centre of circle $T = (2, 3)$ and $P(-2, 5)$

Gradient of $PT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{5 - 3}{-2 - 2} = -\dfrac{1}{2}$

Gradient of the tangent is 2

$$y - y_1 = m(x - x_1)$$
$$y - 5 = 2(x + 2)$$
$$y = 2x + 9$$

Gradient of $QT = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{7 - 3}{4 - 2} = 2$

Gradient of the tangent is $-\dfrac{1}{2}$

$$y - y_1 = m(x - x_1)$$
$$y - 7 = -\frac{1}{2}(x - 4)$$
$$y = -\frac{1}{2}x + 9$$

11 c Gradient $PQ = \frac{7-5}{4-(-2)} = \frac{1}{3}$

Midpoint of $PQ = \left(\frac{-2+4}{2}, \frac{5+7}{2}\right) = (1, 6)$

Gradient of the perpendicular bisector is −3

$y - y_1 = m(x - x_1)$

$y - 6 = -3(x - 1)$

$y = -3x + 9$

d $y = 2x + 9$, $y = -\frac{1}{2}x + 9$ and $y = -3x + 9$

Solve $y = 2x + 9$ and $y = -\frac{1}{2}x + 9$ simultaneously

$2x + 9 = -\frac{1}{2}x + 9$

$4x + 18 = -x + 18$

$x = 0$, $y = 9$

(0, 9)

Solve $y = 2x + 9$ and $y = -3x + 9$ simultaneously

$2x + 9 = -3x + 9$

$x = 0$, $y = 9$

(0, 9)

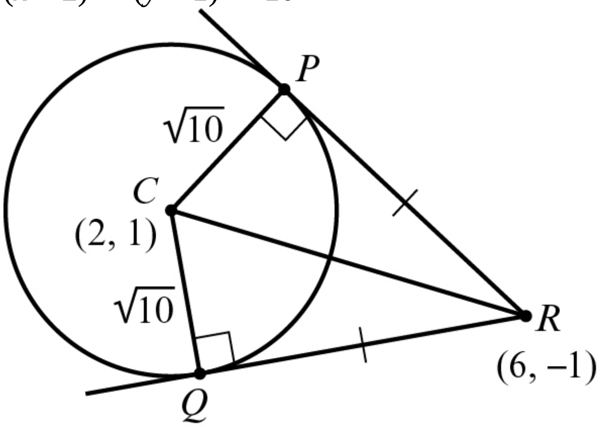
Therefore all three lines intersect at (0, 9)

Challenge

1 y-intercept = −2 so $y = mx - 2$
Substitute $y = mx - 2$ into $(x-7)^2 + (y+1)^2 = 5$
$$(x-7)^2 + (mx-1)^2 = 5$$
$$x^2 - 14x + 49 + m^2x^2 - 2mx + 1 - 5 = 0$$
$$(1+m^2)x^2 - (14+2m)x + 45 = 0$$
Using the discriminant when there are only one root
$$b^2 - 4ac = 0$$
$$(-(14+2m))^2 - 4(1+m^2)(45) = 0$$
$$4m^2 + 56m + 196 - 180 - 180m^2 = 0$$
$$176m^2 - 56m - 16 = 0$$
$$22m^2 - 7m - 2 = 0$$
$$(11m+2)(2m-1) = 0$$
$$m = -\frac{2}{11} \text{ or } m = \frac{1}{2}$$
As m is positive, $m = \frac{1}{2}$

Therefore the equation of the line is $y = \frac{1}{2}x - 2$

2 a $(x-2)^2 + (y-1)^2 = 10$



Radius = $\sqrt{10}$

$CR = \sqrt{(6-2)^2 + (-1-1)^2} = \sqrt{4^2 + (-2)^2} = \sqrt{20}$

Using Pythagoras' theorem, $PR = \sqrt{(\sqrt{20})^2 - (\sqrt{10})^2} = \sqrt{20-10} = \sqrt{10}$

$QR = \sqrt{(\sqrt{20})^2 - (\sqrt{10})^2} = \sqrt{20-10} = \sqrt{10}$

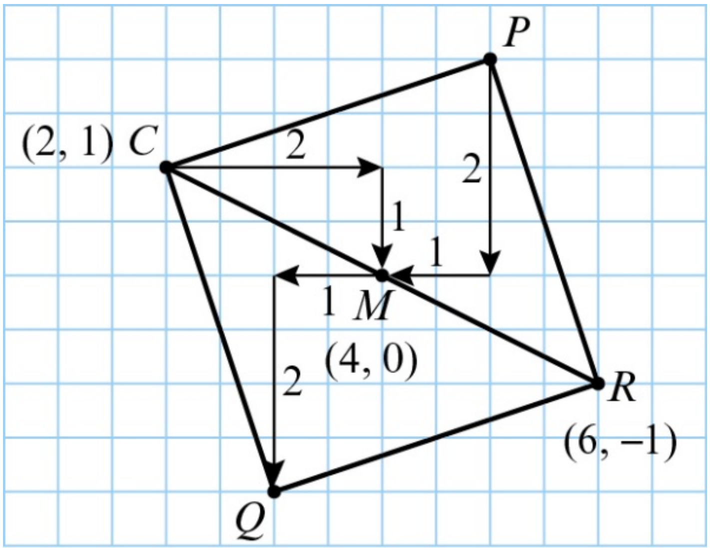
$CP = PR = QR = CQ = \sqrt{10}$, so all four sides of the quadrilateral are the same length.

Using circle theorems, angle CPR = angle CQR = 90° (A radius meets a tangent at 90°.)

2 b Since $QR = CQ$ the angles QCR and QRC are equal and are each 45 degrees. The same is true for angles CRP and RCP. Therefore all the angles at Q, P, C and R are 90°. Therefore, $CPQR$ is a square.

$CPQR$ is a square so it's diagonals bisect at right angles and are equal.

Midpoint of square = midpoint of $CR = \left(\frac{2+6}{2}, \frac{1-1}{2}\right) = (4, 0)$



$P(5, 2)$ and $Q(3, -2)$

Gradient of $PR = \frac{-1-2}{6-5} = -3$

$R(6, -1)$, $m = -3$

$y - y_1 = m(x - x_1)$

$y + 1 = -3(x - 6)$

$y = -3x + 17$

QR is perpendicular to PR, so gradient of $QR = \frac{1}{3}$

$R(6, -1)$, $m = \frac{1}{3}$

$y - y_1 = m(x - x_1)$

$y + 1 = \frac{1}{3}(x - 6)$

$y = \frac{1}{3}x - 3$

The equations are $y = -3x + 17$ and $y = \frac{1}{3}x - 3$