

Circles 6C

1 a $(x_1, y_1) = (3, 2), r = 4$

$$\text{So } (x-3)^2 + (y-2)^2 = 4^2$$

$$(x-3)^2 + (y-2)^2 = 16$$

b $(x_1, y_1) = (-4, 5), r = 6$

$$\text{So } (x-(-4))^2 + (y-5)^2 = 6^2$$

$$(x+4)^2 + (y-5)^2 = 36$$

c $(x_1, y_1) = (5, -6), r = 2\sqrt{3}$

$$\text{So } (x-5)^2 + (y-(-6))^2 = (2\sqrt{3})^2$$

$$(x-5)^2 + (y+6)^2 = 2^2 (\sqrt{3})^2$$

$$(x-5)^2 + (y+6)^2 = 4 \times 3$$

$$(x-5)^2 + (y+6)^2 = 12$$

d $(x_1, y_1) = (2a, 7a), r = 5a$

$$\text{So } (x-2a)^2 + (y-7a)^2 = (5a)^2$$

$$(x-2a)^2 + (y-7a)^2 = 25a^2$$

e $(x_1, y_1) = (-2\sqrt{2}, -3\sqrt{2}), r = 1$

$$\text{So } (x-(-2\sqrt{2}))^2 + (y-(-3\sqrt{2}))^2 = 1^2$$

$$(x+2\sqrt{2})^2 + (y+3\sqrt{2})^2 = 1$$

2 a $(x+5)^2 + (y-4)^2 = 9^2$

$$(x-(-5))^2 + (y-4)^2 = 9^2$$

The centre of the circle is $(-5, 4)$ and the radius is 9.

b $(x-7)^2 + (y-1)^2 = 16$

$$(x-7)^2 + (y-1)^2 = 4^2$$

The centre of the circle is $(7, 1)$ and the radius is 4.

2 c $(x+4)^2 + y^2 = 25$

$$(x - (-4))^2 + (y - 0)^2 = 5^2$$

The centre of the circle is $(-4, 0)$ and the radius is 5.

d $(x+4a)^2 + (y+a)^2 = 144a^2$

$$(x - (-4a))^2 + (y - (-a))^2 = (12a)^2$$

The centre of the circle is $(-4a, -a)$ and the radius is $12a$.

e $(x-3\sqrt{5})^2 + (y+\sqrt{5})^2 = 27$

$$(x-3\sqrt{5})^2 + (y-(-\sqrt{5}))^2 = (\sqrt{27})^2$$

$$\text{Now } \sqrt{27} = \sqrt{9 \times 3} = \sqrt{9} \times \sqrt{3} = 3\sqrt{3}$$

The centre of the circle is $(3\sqrt{5}, -\sqrt{5})$ and the radius is $3\sqrt{3}$.

3 a Substitute $x = 4, y = 8$ into $(x-2)^2 + (y-5)^2 = 13$

$$(x-2)^2 + (y-5)^2 = (4-2)^2 + (8-5)^2 = 2^2 + 3^2 = 4 + 9 = 13 \checkmark$$

So the circle passes through $(4, 8)$.

b Substitute $x = 0, y = -2$ into $(x+7)^2 + (y-2)^2 = 65$

$$(x+7)^2 + (y-2)^2 = (0+7)^2 + (-2-2)^2 = 7^2 + (-4)^2 = 49 + 16 = 65 \checkmark$$

So the circle passes through $(0, -2)$.

c Substitute $x = 7, y = -24$ into $x^2 + y^2 = 25^2$

$$x^2 + y^2 = 7^2 + (-24)^2 = 49 + 576 = 625 = 25^2 \checkmark$$

So the circle passes through $(7, -24)$.

d Substitute $x = 6a, y = -3a$ into $(x-2a)^2 + (y+5a)^2 = 20a^2$

$$(x-2a)^2 + (y+5a)^2 = (6a-2a)^2 + (-3a+5a)^2 = (4a)^2 + (2a)^2 = 16a^2 + 4a^2 = 20a^2 \checkmark$$

So the circle passes through $(6a, -3a)$.

e Substitute $x = \sqrt{5}, y = -\sqrt{5}$ into $(x-3\sqrt{5})^2 + (y-\sqrt{5})^2 = (2\sqrt{10})^2$

$$(x-3\sqrt{5})^2 + (y-\sqrt{5})^2 = (\sqrt{5}-3\sqrt{5})^2 + (-\sqrt{5}-\sqrt{5})^2 = (-2\sqrt{5})^2 + (-2\sqrt{5})^2$$

$$= 4 \times 5 + 4 \times 5 = 20 + 20 = 40 = (\sqrt{40})^2$$

$$\text{Now } \sqrt{40} = \sqrt{4 \times 10} = \sqrt{4} \times \sqrt{10} = 2\sqrt{10} \checkmark$$

So the circle passes through $(\sqrt{5}, -\sqrt{5})$.

- 4 The radius of the circle is

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{(4 - 8)^2 + ((-2) - 1)^2} \\ &= \sqrt{4^2 + 3^2} \\ &= \sqrt{16 + 9} \\ &= \sqrt{25} \\ &= 5\end{aligned}$$

The centre of the circle is (8, 1) and the radius is 5.

$$\text{So } (x - 8)^2 + (y - 1)^2 = 5^2$$

$$\text{or } (x - 8)^2 + (y - 1)^2 = 25$$

- 5 $P(5, 6)$ and $Q(-2, 2)$

The centre of the circle is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{5 + (-2)}{2}, \frac{6 + 2}{2}\right) = \left(\frac{3}{2}, \frac{8}{2}\right) = \left(\frac{3}{2}, 4\right)$$

The radius of the circle is

$$\begin{aligned}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} &= \sqrt{\left(5 - \frac{3}{2}\right)^2 + (6 - 4)^2} \\ &= \sqrt{\left(\frac{7}{2}\right)^2 + (2)^2} \\ &= \sqrt{\frac{49}{4} + 4} \\ &= \sqrt{\frac{49}{4} + \frac{16}{4}} \\ &= \sqrt{\frac{65}{4}}\end{aligned}$$

So the equation of the circle is

$$\left(x - \frac{3}{2}\right)^2 + (y - 4)^2 = \left(\sqrt{\frac{65}{4}}\right)^2 \quad \text{or} \quad \left(x - \frac{3}{2}\right)^2 + (y - 4)^2 = \frac{65}{4}$$

- 6 Substitute $x = 1$, $y = -3$ into $(x - 3)^2 + (y + 4)^2 = r^2$

$$(1 - 3)^2 + (-3 + 4)^2 = r^2$$

$$(-2)^2 + (1)^2 = r^2$$

$$5 = r^2$$

$$\text{So } r = \sqrt{5}$$

7 a Substitute (2,2) into $(x-2)^2 + (y-4)^2 = r^2$

$$(2-2)^2 + (2-4)^2 = r^2$$

$$0^2 + (-2)^2 = r^2$$

$$r^2 = 4$$

$$r = 2$$

b The distance between (2, 2) and $(2 + \sqrt{3}, 5)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2 + \sqrt{3} - 2)^2 + (5 - 2)^2}$$

$$= \sqrt{(\sqrt{3})^2 + 3^2}$$

$$= \sqrt{3 + 9}$$

$$= \sqrt{12}$$

The distance between (2, 2) and $(2 - \sqrt{3}, 5)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{(2 - \sqrt{3} - 2)^2 + (5 - 2)^2}$$

$$= \sqrt{(-\sqrt{3})^2 + (3)^2}$$

$$= \sqrt{3 + 9}$$

$$= \sqrt{12}$$

The distance between $(2 + \sqrt{3}, 5)$ and $(2 - \sqrt{3}, 5)$ is

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} = \sqrt{((2 - \sqrt{3}) - (2 + \sqrt{3}))^2 + (5 - 5)^2}$$

$$= \sqrt{(2 - \sqrt{3} - 2 - \sqrt{3})^2 + 0^2}$$

$$= \sqrt{(-2\sqrt{3})^2}$$

$$= \sqrt{4 \times 3}$$

$$= \sqrt{12}$$

PQ , QR and PR all equal $\sqrt{12}$.

So $\triangle PQR$ is equilateral.

8 a Rearrange $x^2 + y^2 - 4x - 11 = 0$ into the form $(x - a)^2 + y^2 = r^2$

$$x^2 - 4x - 11 + y^2 = 0$$

Completing the square gives

$$(x - 2)^2 - 4 - 11 + y^2 = 0$$

$$(x - 2)^2 + y^2 = 15$$

b Centre of the circle = $(2, 0)$, radius = $\sqrt{15}$

9 a $x^2 + y^2 - 10x + 4y - 20 = 0$

$$x^2 - 10x + y^2 + 4y - 20 = 0$$

Completing the square gives

$$(x - 5)^2 - 25 + (y + 2)^2 - 4 - 20 = 0$$

$$(x - 5)^2 + (y + 2)^2 = 49$$

b Centre of the circle = $(5, -2)$, radius = 7

10 a $x^2 + y^2 - 2x + 8y - 8 = 0$

$$x^2 - 2x + y^2 + 8y - 8 = 0$$

Completing the square gives

$$(x - 1)^2 - 1 + (y + 4)^2 - 16 - 8 = 0$$

$$(x - 1)^2 + (y + 4)^2 = 25$$

Centre of the circle = $(1, -4)$, radius = 5

b $x^2 + y^2 + 12x - 4y = 9$

$$x^2 + 12x + y^2 - 4y = 9$$

Completing the square gives

$$(x + 6)^2 - 36 + (y - 2)^2 - 4 = 9$$

$$(x + 6)^2 + (y - 2)^2 = 49$$

Centre of the circle = $(-6, 2)$, radius = 7

c $x^2 + y^2 - 6y = 22x - 40$

$$x^2 - 22x + y^2 - 6y = -40$$

Completing the square gives

$$(x - 11)^2 - 121 + (y - 3)^2 - 9 = -40$$

$$(x - 11)^2 + (y - 3)^2 = 90$$

Centre of the circle = $(11, 3)$, radius = $\sqrt{90} = 3\sqrt{10}$

d $x^2 + y^2 + 5x - y + 4 = 2y + 8$

$$x^2 + 5x + y^2 - 3y = 4$$

Completing the square gives

$$\left(x + \frac{5}{2}\right)^2 - \frac{25}{4} + \left(y - \frac{3}{2}\right)^2 - \frac{9}{4} = 4$$

$$10 \text{ d } \left(x + \frac{5}{2}\right)^2 + \left(y - \frac{3}{2}\right)^2 = \frac{25}{2}$$

$$\text{Centre of the circle} = \left(-\frac{5}{2}, \frac{3}{2}\right), \text{ radius} = \frac{5}{\sqrt{2}} = \frac{5\sqrt{2}}{2}$$

$$e \quad 2x^2 + 2y^2 - 6x + 5y = 2x - 3y - 3$$

$$2x^2 - 8x + 2y^2 + 8y = -3$$

Completing the square gives

$$2((x - 2)^2 - 4) + 2((y + 2)^2 - 4) = -3$$

$$2(x - 2)^2 - 8 + 2(y + 2)^2 - 8 = -3$$

$$2(x - 2)^2 + 2(y + 2)^2 = 13$$

$$(x - 2)^2 + (y + 2)^2 = \frac{13}{2}$$

$$\text{Centre of the circle} = (2, -2), \text{ radius} = \sqrt{\frac{13}{2}} = \frac{\sqrt{26}}{2}$$

$$11 \text{ a } x^2 + y^2 + 12x + 2y = k$$

$$x^2 + 12x + y^2 + 2y = k$$

$$(x + 6)^2 - 36 + (y + 1)^2 - 1 = k$$

$$(x + 6)^2 + (y + 1)^2 = k + 37$$

$$\text{Centre of the circle} = (-6, -1)$$

b A circle must have a positive radius, so $k + 37 > 0$

$$\text{So } k > -37$$

$$12 \quad x^2 + y^2 + 6x - 14y = 483$$

$$x^2 + 6x + y^2 - 14y = 483$$

Completing the square gives

$$(x + 3)^2 - 9 + (y - 7)^2 - 49 = 483$$

$$(x + 3)^2 + (y - 7)^2 = 541$$

$$\text{Centre of the circle} = (-3, 7), \text{ radius} = \sqrt{541}$$

$$P(7, -14), \text{ midpoint of the line} = (-3, 7)$$

$$\text{Midpoint} = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(-3, 7) = \left(\frac{7 + x_2}{2}, \frac{-14 + y_2}{2}\right)$$

$$-3 = \frac{7 + x_2}{2} \quad \text{and} \quad 7 = \frac{-14 + y_2}{2}$$

$$x_1 = -13, x_2 = 28$$

$$Q(-13, 28)$$

- 13** $(x - k)^2 + y^2 = 41$, (3, 4)
 Substitute $x = 3$ and $y = 4$ into the equation $(x - k)^2 + y^2 = 41$
 $(3 - k)^2 + 4^2 = 41$
 $k^2 - 6k + 9 + 16 = 41$
 $k^2 - 6k - 16 = 0$
 $(k + 2)(k - 8) = 0$
 $k = -2$ or $k = 8$

Challenge

- 1** $(x - k)^2 + (y - 2)^2 = 50$, (4, -5)
 Substitute $x = 4$ and $y = -5$ into the equation $(x - k)^2 + (y - 2)^2 = 50$
 $(4 - k)^2 + (-5 - 2)^2 = 50$
 $k^2 - 8k + 16 + 49 = 50$
 $k^2 - 8k + 15 = 0$
 $(k - 3)(k - 5) = 0$
 $k = 3$ or $k = 5$
 $(x - 3)^2 + (y - 2)^2 = 50$ or $(x - 5)^2 + (y - 2)^2 = 50$

- 2** $x^2 + y^2 + 2fx + 2gy + c = 0$
 Rearranging the equation:
 $x^2 + 2fx + y^2 + 2gy + c = 0$
 Completing the square gives
 $(x + f)^2 - f^2 + (y + g)^2 - g^2 + c = 0$
 $(x + f)^2 + (y + g)^2 = f^2 + g^2 - c$
 The centre of the circle is $(-f, -g)$ and the radius is $\sqrt{f^2 + g^2 - c}$.