Circles 6A

1 a $(x_1, y_1) = (4, 2), (x_2, y_2) = (6, 8)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{4+6}{2}, \frac{2+8}{2}\right) = \left(\frac{10}{2}, \frac{10}{2}\right) = (5, 5)$

b $(x_1, y_1) = (0, 6), (x_2, y_2) = (12, 2)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{0+12}{2}, \frac{6+2}{2}\right) = \left(\frac{12}{2}, \frac{8}{2}\right) = (6, 4)$

c $(x_1, y_1) = (2, 2), (x_2, y_2) = (-4, 6)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{2+(-4)}{2}, \frac{2+6}{2}\right) = \left(\frac{-2}{2}, \frac{8}{2}\right) = (-1, 4)$

d $(x_1, y_1) = (-6, 4), (x_2, y_2) = (6, -4)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-6+6}{2}, \frac{4+(-4)}{2}\right) = \left(\frac{0}{2}, \frac{0}{2}\right) = (0, 0)$

e $(x_1, y_1) = (7, -4), (x_2, y_2) = (-3, 6)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{7+(-3)}{2}, \frac{-4+6}{2}\right) = \left(\frac{4}{2}, \frac{2}{2}\right) = (2, 1)$

f $(x_1, y_1) = (-5, -5), (x_2, y_2) = (-11, 8)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-5+(-11)}{2}, \frac{-5+8}{2}\right) = \left(\frac{-16}{2}, \frac{3}{2}\right) = \left(-8, \frac{3}{2}\right)$

g $(x_1, y_1) = (6a, 4b), (x_2, y_2) = (2a, -4b)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{6a+2a}{2}, \frac{4b+(-4b)}{2}\right) = \left(\frac{8a}{2}, \frac{0}{2}\right) = (4a, 0)$

h $(x_1, y_1) = (-4u, 0), (x_2, y_2) = (3u, -2v)$

So $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right) = \left(\frac{-4u+3u}{2}, \frac{0+(-2v)}{2}\right) = \left(\frac{-u}{2}, -v\right)$

1 i $(x_1, y_1) = (a+b, 2a-b), (x_2, y_2) = (3a-b, -b)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{a+b+3a-b}{2}, \frac{2a-b+(-b)}{2}\right) = \left(\frac{4a}{2}, \frac{2a-2b}{2}\right) = (2a, a-b)$

j $(x_1, y_1) = (4\sqrt{2}, 1), (x_2, y_2) = (2\sqrt{2}, 7)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{4\sqrt{2}+2\sqrt{2}}{2}, \frac{1+7}{2}\right) = \left(\frac{6\sqrt{2}}{2}, \frac{8}{2}\right) = (3\sqrt{2}, 4)$

k $(x_1, y_1) = (\sqrt{2}-\sqrt{3}, 3\sqrt{2}+4\sqrt{3}), (x_2, y_2) = (3\sqrt{2}+\sqrt{3}, -\sqrt{2}+2\sqrt{3})$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\sqrt{2}-\sqrt{3}+3\sqrt{2}+\sqrt{3}}{2}, \frac{3\sqrt{2}+4\sqrt{3}+(-\sqrt{2}+2\sqrt{3})}{2}\right)$

$= \left(\frac{4\sqrt{2}}{2}, \frac{2\sqrt{2}+6\sqrt{3}}{2}\right)$

$= (2\sqrt{2}, \sqrt{2}+3\sqrt{3})$

2 A(−2, 5) and B(a, b), midpoint M(4, 3)

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

(4, 3) = $\left(\frac{-2+a}{2}, \frac{5+b}{2}\right)$

4 = $\frac{-2+a}{2}$ and 3 = $\frac{5+b}{2}$

a = 10 and b = 1

3 $(x_1, y_1) = (-4, 6), (x_2, y_2) = (7, 8)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-4+7}{2}, \frac{6+8}{2}\right) = \left(\frac{3}{2}, \frac{14}{2}\right) = \left(\frac{3}{2}, 7\right)$

The centre is $\left(\frac{3}{2}, 7\right)$.

4 $(x_1, y_1) = \left(\frac{4a}{5}, \frac{-3b}{4}\right), (x_2, y_2) = \left(\frac{2a}{5}, \frac{5b}{4}\right)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\frac{4a}{5}+\frac{2a}{5}}{2}, \frac{\frac{-3b}{4}+\frac{5b}{4}}{2}\right) = \left(\frac{\frac{6a}{5}}{2}, \frac{\frac{2b}{4}}{2}\right) = \left(\frac{3a}{5}, \frac{b}{4}\right)$

4 The centre is $\left(\frac{3a}{5}, \frac{b}{4}\right)$.

5 a A(−3, −4) and B(6, 10)

centre of circle = midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{-3+6}{2}, \frac{-4+10}{2}\right) = \left(\frac{3}{2}, 3\right)$

b $y = 2x$

When $x = \frac{3}{2}$, $y = 2(\frac{3}{2}) = 3$, therefore the centre of the circle $(\frac{3}{2}, 3)$ lies on the line $y = 2x$.

6 $J\left(\frac{3}{4}, \frac{4}{3}\right)$ and $K\left(-\frac{1}{2}, 2\right)$

centre of circle = midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{\frac{3}{4}+\left(-\frac{1}{2}\right)}{2}, \frac{\frac{4}{3}+2}{2}\right) = \left(\frac{1}{8}, \frac{5}{3}\right)$

$y = 8x + b$

At $\left(\frac{1}{8}, \frac{5}{3}\right)$, $\frac{5}{3} = 8\left(\frac{1}{8}\right) + b$

$b = \frac{2}{3}$

7 $(x_1, y_1) = (0, -2), (x_2, y_2) = (6, -5)$

So $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right) = \left(\frac{0+6}{2}, \frac{-2+(-5)}{2}\right) = \left(\frac{6}{2}, \frac{-7}{2}\right) = \left(3, -\frac{7}{2}\right)$

Substitute $x = 3$ and $y = -\frac{7}{2}$ into $x - 2y - 10 = 0$:

$(3) - 2\left(-\frac{7}{2}\right) - 10 = 3 + 7 - 10 = 0$

So the centre is on the line $x - 2y - 10 = 0$.

8 $(x_1, y_1) = (a, b), (x_2, y_2) = (2, -3)$

The centre is (6, 1) so

$\left(\frac{a+2}{2}, \frac{b+(-3)}{2}\right) = (6, 1)$

$\frac{a+2}{2} = 6$

8 $a+2=12$

$a=10$

$$\frac{b+(-3)}{2}=1$$

$$\frac{b-3}{2}=1$$

$b-3=2$

$b=5$

The coordinates of *G* are (10, 5).

9 $(x_1, y_1)=(p, q),(x_2, y_2)=(3a, -7a)$

The centre is (–2*a*, 5*a*) so

$$\left(\frac{p+3a}{2}, \frac{q+(-7a)}{2}\right)=(-2a, 5a)$$

$$\frac{p+3a}{2}=-2a$$

$p+3a=-4a$

$p=-7a$

$$\frac{q+(-7a)}{2}=5a$$

$$\frac{q-7a}{2}=5a$$

$q-7a=10a$

$q=17a$

The coordinates of *C* are (–7*a*, 17*a*).

10 $(x_1, y_1)=(3, p),(x_2, y_2)=(q, 4)$ so

$$\left(\frac{3+q}{2}, \frac{p+4}{2}\right)=(5, 6)$$

$$\frac{3+q}{2}=5$$

$3+q=10$

$q=7$

$$\frac{p+4}{2}=6$$

$p+4=12$

$p=8$

so $p=8, q=7$

11 $(x_1, y_1) = (-4, 2a), (x_2, y_2) = (3b, -4)$ so

$$\left(\frac{-4+3b}{2}, \frac{2a-4}{2}\right) = (b, 2a)$$

$$\frac{-4+3b}{2} = b$$

$$-4+3b = 2b$$

$$-4 = -b$$

$$b = 4$$

$$\frac{2a-4}{2} = 2a$$

$$2a-4 = 4a$$

$$-4 = 2a$$

$$a = -2$$

so $a = -2, b = 4$

Challenge

a $B(7, 11)$ and $C(p, q)$. Midpoint of BC is $M(8, 5)$

Midpoint = $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$

$(8, 5) = \left(\frac{7+p}{2}, \frac{11+q}{2}\right)$

$8 = \frac{7+p}{2}$ and $5 = \frac{11+q}{2}$

$p = 9$ and $q = -1$

b $A(3, 5)$ and $B(7, 11)$

Midpoint AB = $= \left(\frac{3+7}{2}, \frac{5+11}{2}\right) = (5, 8)$

Equation of line joining (5, 8) and $M(8, 5)$:

$$\frac{y-8}{5-8} = \frac{x-5}{8-5}$$

$$y-8 = -(x-5)$$

$$y = -x+13$$

c Gradient of line AC = $\frac{y_2-y_1}{x_2-x_1} = \frac{(-1)-5}{9-3} = -1$

The gradient of the line $y = -x + 13$ is -1. The two gradients are equal so the two lines are parallel.