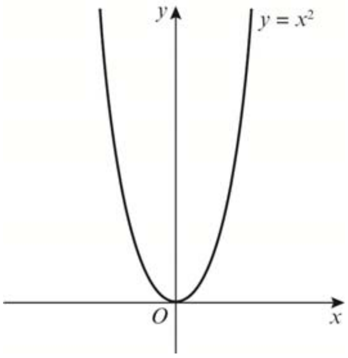
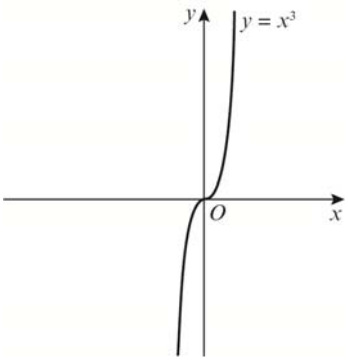
Graphs and transformations 4E

1 Sketches of original graphs:

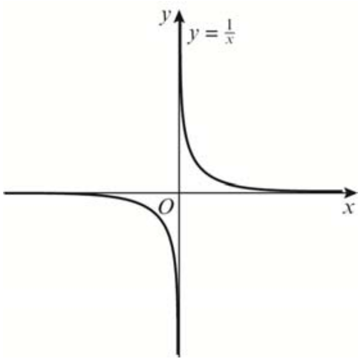
$f(x) = x^2$



$f(x) = x^3$

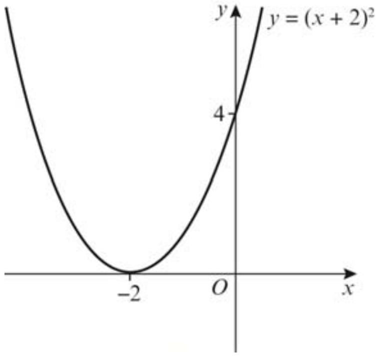


$f(x) = \frac{1}{x}$



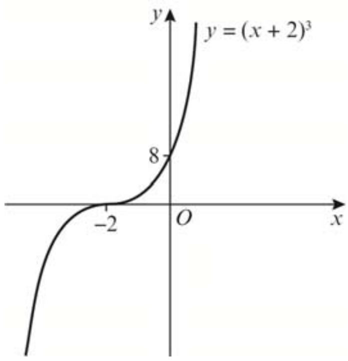
a $f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.

a i $f(x) = x^2$, $f(x + 2) = (x + 2)^2$



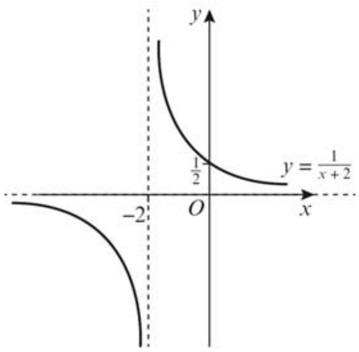
The curve touches the x-axis at $(-2, 0)$ and crosses the y-axis at $(0, 4)$.

ii $f(x) = x^3$, $f(x + 2) = (x + 2)^3$



The curve crosses the x-axis at $(-2, 0)$ and crosses the y-axis at $(0, 8)$.

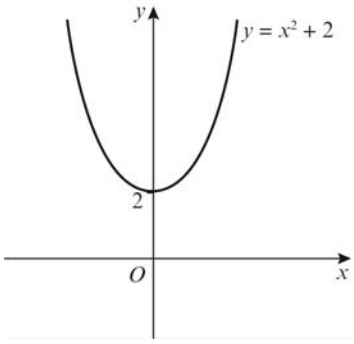
iii $f(x) = \frac{1}{x}$, $f(x + 2) = \frac{1}{x+2}$



The curve crosses the y-axis at $(0, \frac{1}{2})$.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = -2$.

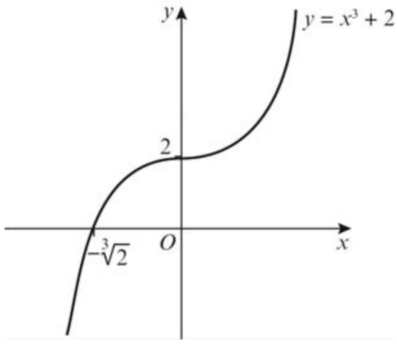
1 b f(x) + 2 is a translation of the graph of f(x) by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.

i $f(x) = x^2$, $f(x) + 2 = x^2 + 2$



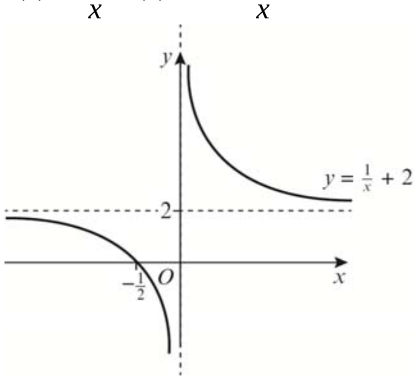
The curve crosses the y-axis at (0, 2).

ii $f(x) = x^3$, $f(x) + 2 = x^3 + 2$



The curve crosses the x-axis at $\left(-\sqrt[3]{2}, 0\right)$ and crosses the y-axis at (0, 2).

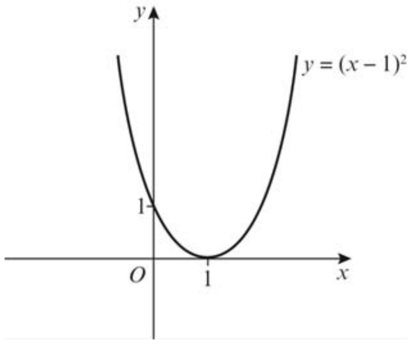
iii $f(x) = \frac{1}{x}$, $f(x) + 2 = \frac{1}{x} + 2$



The horizontal asymptote is $y = 2$.
The vertical asymptote is $x = 0$.

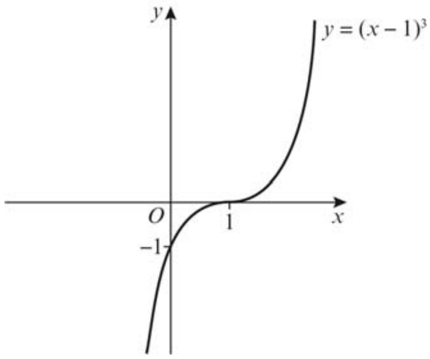
c f(x − 1) is a translation of the graph of f(x) by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, or one unit to the right.

i $f(x) = x^2$, $f(x - 1) = (x - 1)^2$



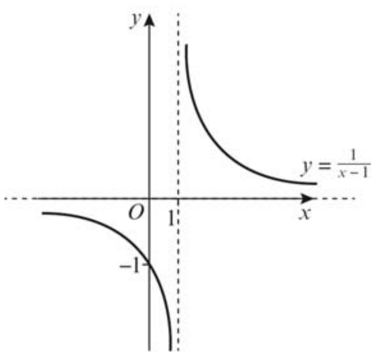
The curve touches the x-axis at (1, 0) and crosses the y-axis at (0, 1).

ii $f(x) = x^3$, $f(x - 1) = (x - 1)^3$



The curve crosses the x-axis at (1, 0) and crosses the y-axis at (0, −1).

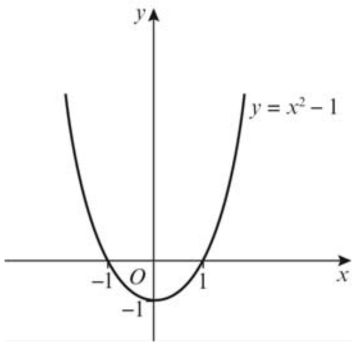
iii $f(x) = \frac{1}{x}$, $f(x - 1) = \frac{1}{x-1}$



The curve crosses the y-axis at (0, −1).
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 1$.

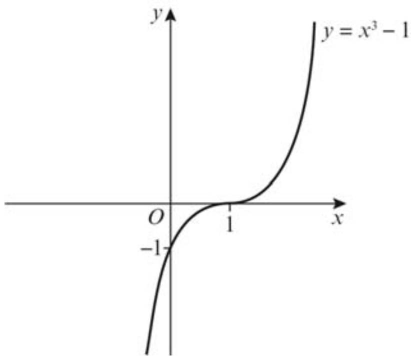
1 d f(x) − 1 is a translation of the graph of f(x) by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit down.

i $f(x) = x^2$, $f(x) - 1 = x^2 - 1$



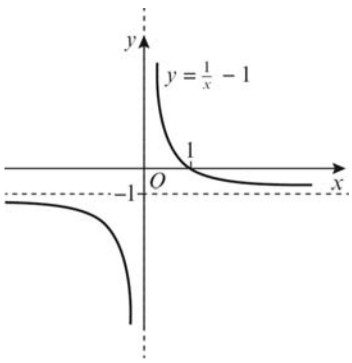
The curve crosses the x-axis at (−1, 0) and (1, 0) and crosses the y-axis at (0, −1).

ii $f(x) = x^3$, $f(x) - 1 = x^3 - 1$



The curve crosses the x-axis at (1, 0) and crosses the y-axis at (0, −1).

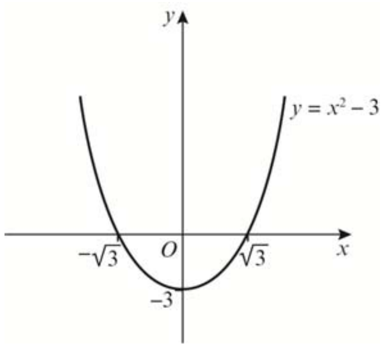
iii $f(x) = \frac{1}{x}$, $f(x) - 1 = \frac{1}{x} - 1$



1 d iii The curve crosses the x-axis at (1, 0).
The horizontal asymptote is $y = -1$.
The vertical asymptote is $x = 0$.

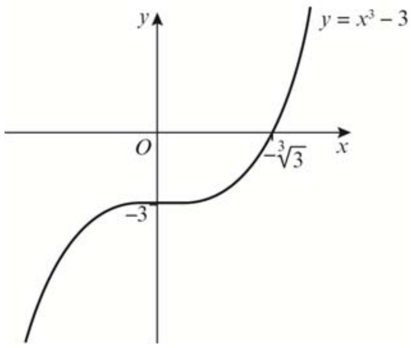
e f(x) − 3 is a translation of the graph of f(x) by $\begin{pmatrix} 0 \\ -3 \end{pmatrix}$, or three units down.

i $f(x) = x^2$, $f(x) - 3 = x^2 - 3$



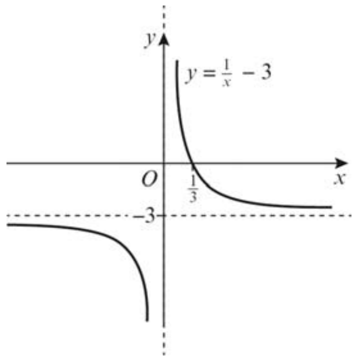
The curve crosses the x-axis at $\left(-\sqrt{3}, 0\right)$ and $\left(\sqrt{3}, 0\right)$ and crosses the y-axis at (0, −3).

ii $f(x) = x^3$, $f(x) - 3 = x^3 - 3$



The curve crosses the x-axis at $\left(-\sqrt[3]{3}, 0\right)$ and crosses the y-axis at (0, −3).

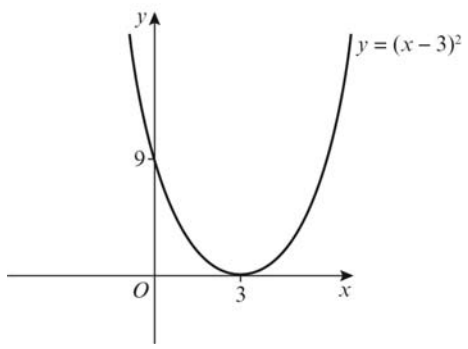
1 e iii $f(x) = \dfrac{1}{x}$, $f(x) - 3 = \dfrac{1}{x} - 3$



The curve crosses the x-axis at $\left(\frac{1}{3}, 0\right)$.
The horizontal asymptote is $y = -3$.
The vertical asymptote is $x = 0$.

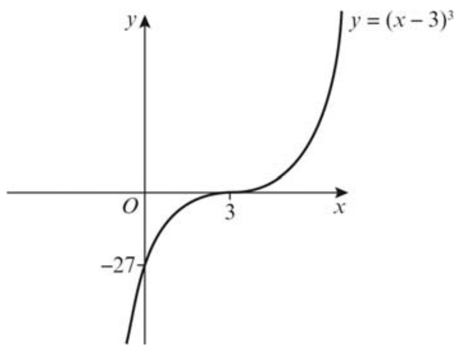
f $f(x - 3)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 3 \\ 0 \end{pmatrix}$, or three units to the right.

i $f(x) = x^2$, $f(x - 3) = (x - 3)^2$



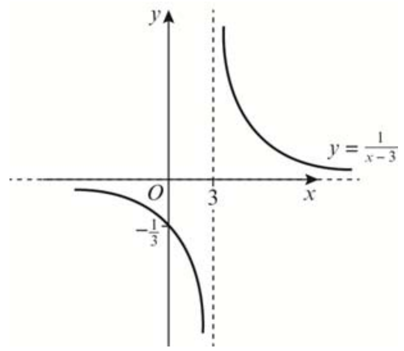
The curve touches the x-axis at (3, 0) and crosses the y-axis at (0, 9).

ii $f(x) = x^3$, $f(x - 3) = (x - 3)^3$



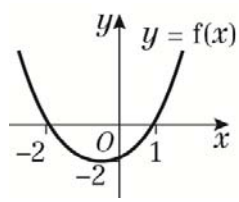
f ii The curve crosses the x-axis at (3,0) and crosses the y-axis at (0, −27).

iii $f(x) = \dfrac{1}{x}$, $f(x - 3) = \dfrac{1}{x - 3}$

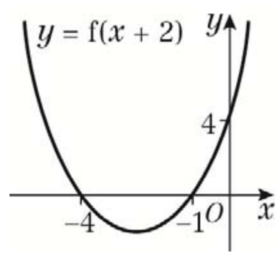


The curve crosses the y-axis at $\left(0, -\frac{1}{3}\right)$.
The horizontal asymptote is $y = 0$.
The vertical asymptote is $x = 3$.

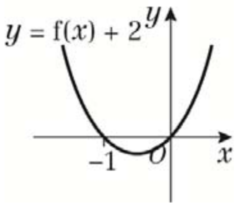
2 a $y = (x - 1)(x + 2)$
As $a = 1$ is positive, the graph has a ∪ shape and a minimum point.
$0 = (x - 1)(x + 2)$
So $x = 1$ or $x = -2$
The curve crosses the x-axis at (1, 0) and (−2, 0).
When $x = 0$, $y = (-1) \times 2 = -2$
The curve crosses the y-axis at (0, −2).



b i $f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.



2 b ii $f(x) + 2$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.

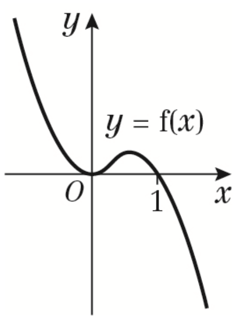


Since axis of symmetry of $f(x)$ is at $x = -\frac{1}{2}$, the same axis of symmetry applies to $f(x) + 2$.
Since one root is at $x = 0$, the other must be symmetric at $x = -1$.

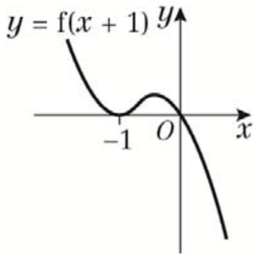
c $y = f(x + 2)$ is
$y = (x + 2 - 1)(x + 2 + 2)$
$\quad = (x + 1)(x + 4)$
When $x = 0$, $y = 4$

$y = f(x) + 2$ is
$y = (x - 1)(x + 2) + 2$
$\quad = x^2 + x - 2 + 2$
$\quad = x^2 + x$
When $x = 0$, $y = 0$

3 a $y = x^2(1 - x)$
$0 = x^2(1 - x)$
So $x = 0$ or $x = 1$
The curve crosses the x-axis at (1, 0) and touches it at (0, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$

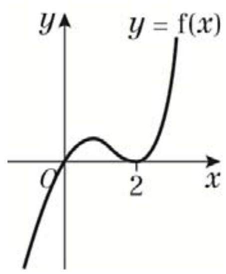


3 b $f(x + 1)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ -1 \end{pmatrix}$, or one unit to the left.

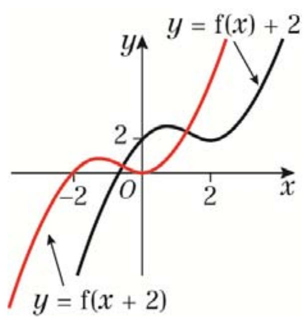


c $f(x + 1) = (x + 1)^2(1 - (x + 1))$
$\quad = -(x + 1)^2 x$
When $x = 0$, $y = 0$; the curve passes through (0, 0).

4 a $y = x(x - 2)^2$
$0 = x(x - 2)^2$
So $x = 0$ or $x = 2$
The curve crosses the x-axis at (0, 0) and touches it at (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$

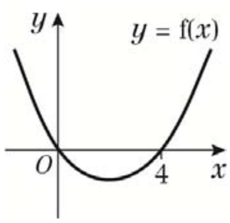


b $f(x) + 2$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$, or two units up.
$f(x + 2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix} -2 \\ 0 \end{pmatrix}$, or two units to the left.

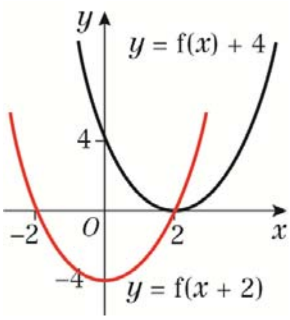


4 c $f(x+2) = (x+2)((x+2)-2)^2$
$= (x+2)x^2$
$(x+2)(x)^2 = 0$
So $x = 0$ and $x = -2$
The graph crosses the axes at (0, 0) and (−2, 0).

5 a $y = x(x-4)$
As $a = 1$ is positive, the graph has a ∪ shape and a minimum point.
$0 = x(x-4)$
So $x = 0$ or $x = 4$
The curve crosses the x-axis at (0, 0) and (4, 0).



b $f(x+2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}-2\\0\end{pmatrix}$, or two units to the left.
$f(x) + 4$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}0\\4\end{pmatrix}$, or four units up.

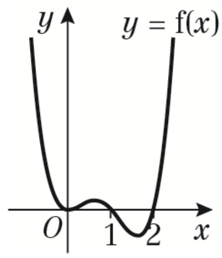


c $f(x+2) = (x+2)((x+2)-4)$
$= (x+2)(x-2)$
$0 = (x+2)(x-2)$
So $x = -2$ or $x = 2$
When $x = 0$, $y = 2 \times (-2) = -4$
So $f(x+2)$ crosses the x-axis at (−2, 0) and (2, 0) and the y-axis at (0, −4).

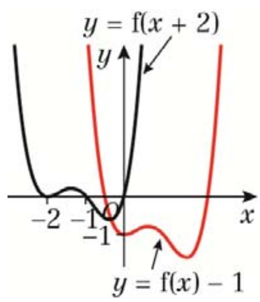
$f(x) + 4 = x(x-4) + 4$
$= x^2 - 4x + 4$
$= (x-2)^2$

5 c $0 = (x-2)^2$
So $x = 2$
When $x = 0$, $y = (-2)^2 = 4$
So $f(x) + 4$ touches the x-axis at (2, 0) and crosses the y-axis at (0, 4).

6 a $y = x^2(x-1)(x-2)$
$0 = x^2(x-1)(x-2)$
So $x = 0$, $x = 1$ or $x = 2$
The curve touches the x-axis at (0, 0) and crosses it at (1, 0) and (2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = f(x+2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}-2\\0\end{pmatrix}$, or two units to the left.
$y = f(x) - 1$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}0\\-1\end{pmatrix}$, or one unit down.



7 a $y = f(x-2)$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}2\\0\end{pmatrix}$, or two units to the right.
So P translates to (6, −1).

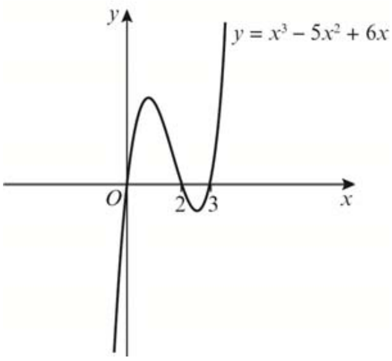
b $y = f(x) + 3$ is a translation of the graph of $f(x)$ by $\begin{pmatrix}0\\3\end{pmatrix}$, or three units up.
So P translates to (4, 2).

8 $y = f(x)$ has asymptotes at $x = 0$ and $y = 0$.
Asymptotes after the translation are at $x = 4$ and $y = 0$, therefore the graph has been translated four units to the right.

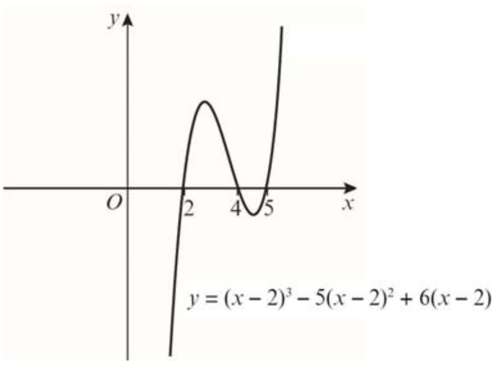
$f(x) = \frac{1}{x}$, $f(x-4) = \frac{1}{x-4}$

$y = \frac{1}{x-4}$

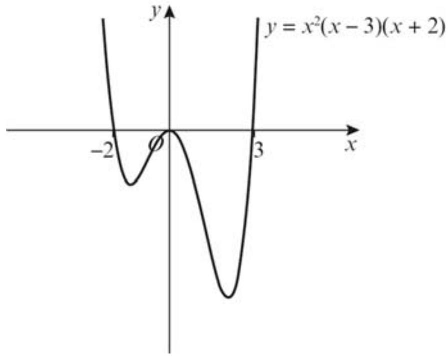
9 a $y = x^3 - 5x^2 + 6x$
$= x(x^2 - 5x + 6)$
$= x(x-2)(x-3)$
$0 = x(x-2)(x-3)$
So $x = 0$, $x = 2$ or $x = 3$
The curve crosses the x-axis at (0, 0), (2, 0) and (3, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



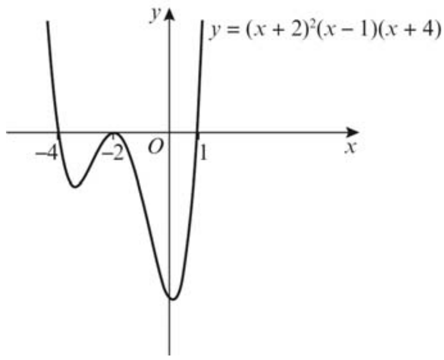
b Let $f(x) = x^3 - 5x^2 + 6x$
$(x-2)^3 - 5(x-2)^2 + 6(x-2)$ is $f(x-2)$, which is a translation of two units to the right.



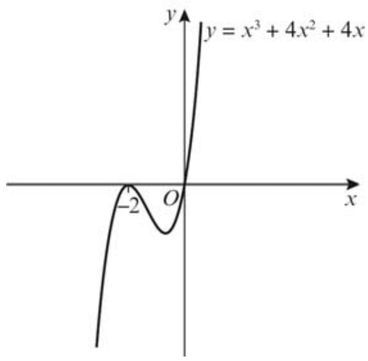
10 a $y = x^2(x-3)(x+2)$
$0 = x^2(x-3)(x+2)$
So $x = 0$, $x = 3$ or $x = -2$
The curve touches the x-axis at (0, 0) and crosses it at (3, 0) and (−2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b Let $f(x) = x^2(x-3)(x+2)$
$(x+2)^2(x-1)(x+4)$ is $f(x+2)$, which is a translation of two units to the left.

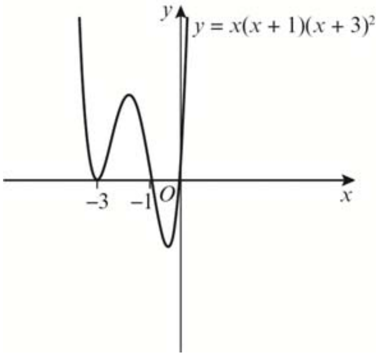


11 a $y = x^3 + 4x^2 + 4x$
$= x(x^2 + 4x + 4)$
$= x(x+2)^2$
So $x = 0$ or $x = -2$
The curve crosses the x-axis at (0, 0) and touches it at (−2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



11 b $y = (x + a)^3 + 4(x + a)^2 + 4(x + a)$
$y = x^3 + 4x^2 + 4x$ crosses the x-axis at (0, 0) and (−2, 0).
So for the point (−1, 0) to lie on the curve, the graph must be translated either one unit to the left or one unit to the right.
$a = -1$ or $a = 1$

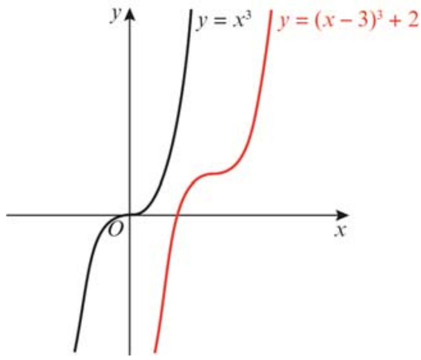
12 a $y = x(x + 1)(x + 3)^2$
$0 = x(x + 1)(x + 3)^2$
So $x = 0$, $x = -1$ or $x = -3$
The curve crosses the x-axis at (0, 0) and (−1, 0) and touches it (−3, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



b $y = (x + b)(x + b + 1)(x + b + 3)^2$
$y = x(x + 1)(x + 3)^2$ crosses the x-axis at (0, 0), (−1, 0) and (−3, 0).
So for the point (2, 0) to lie on the curve, the graph must be translated either two units to the right, three units to the right or five units to the right.
$b = -2$, $b = -3$ or $b = -5$

Challenge

1 The graph of $f(x) = x^3$ has a point of inflection at (0, 0).
$y = (x - 3)^3 + 2$ is a translation of three units to the right and two units up.
The point of inflection is (3, 2).



2 a $y = f(x + 2) - 5$ is a translation by $\begin{pmatrix} -2 \\ -5 \end{pmatrix}$,
or two units to the left and five units down.
So the point $Q(-5, -7)$ is transformed to the point (−7, −12).

b The coordinates of the point $Q(-5, -7)$ is transformed to the point (−3, −6).
This is a translation of two units to the right and one unit up.
So $y = f(x - 2) + 1$