Graphs and transformations 4D

1 a i $y = x^2$ is standard.

$y = x(x^2 - 1)$

$= x(x - 1)(x + 1)$

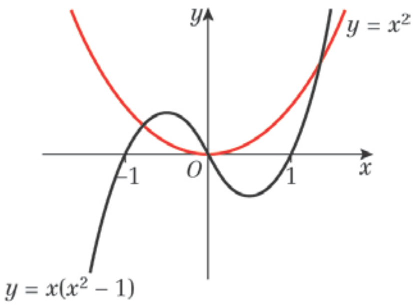
$0 = x(x - 1)(x + 1)$

So $x = 0$, $x = 1$ or $x = -1$

So the curve crosses the x-axis at $(0, 0)$, $(1, 0)$ and $(-1, 0)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



ii Three points of intersection

iii Equation: $x^2 = x(x^2 - 1)$

b i $y = x(x + 2)$

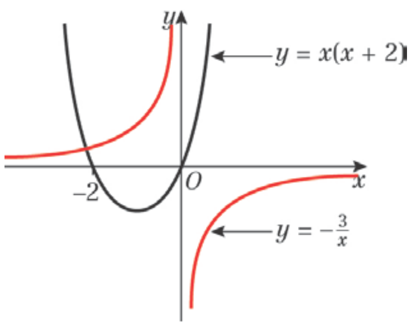
As $a = 1$ is positive, the graph has a $\vee$ shape and a minimum point.

$0 = x(x + 2)$

So $x = 0$ or $x = -2$

So the curve crosses the x-axis at $(0, 0)$ and $(-2, 0)$.

$y = -\dfrac{3}{x}$ is like $y = -\dfrac{1}{x}$ and so exists in the second and fourth quadrants.



ii One point of intersection

iii Equation: $x(x + 2) = -\dfrac{3}{x}$

c i $y = x^2$ is standard.

$y = (x + 1)(x - 1)^2$

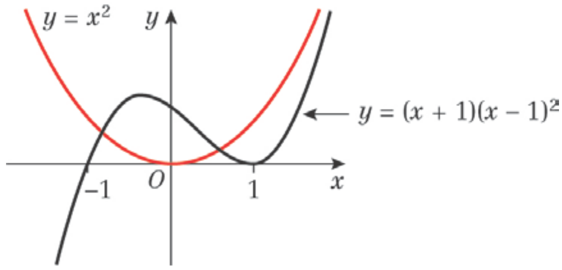
$0 = (x + 1)(x - 1)^2$

So $x = -1$ or $x = 1$

So the curve crosses the x-axis at $(-1, 0)$ and touches it at $(1, 0)$.

$x \to \infty, y \to +\infty$

$x \to -\infty, y \to -\infty$



ii Three points of intersection

iii Equation: $x^2 = (x + 1)(x - 1)^2$

d i $y = x^2(1 - x)$

$0 = x^2(1 - x)$

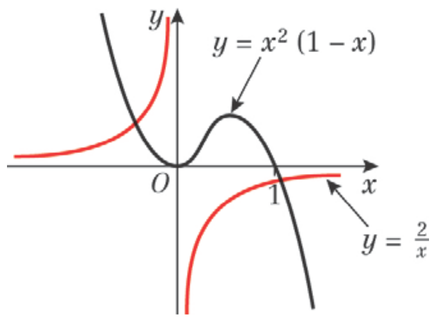
So $x = 0$ or $x = 1$

So the curve crosses the x-axis at $(1, 0)$ and touches it at $(0, 0)$.

$x \to \infty, y \to -\infty$

$x \to -\infty, y \to \infty$

$y = -\dfrac{2}{x}$ is like $y = -\dfrac{1}{x}$ and so exists in the second and fourth quadrants.



ii Two points of intersection

iii Equation: $x^2(1 - x) = -\dfrac{2}{x}$

1 e i $y = x(x-4)$

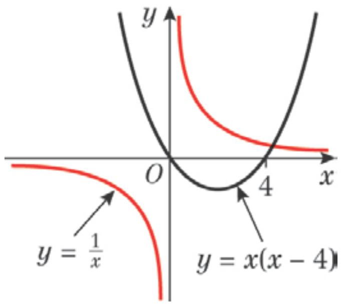
As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

$0 = x(x-4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

$y = \frac{1}{x}$ is standard.



ii One point of intersection

iii Equation: $x(x-4) = \frac{1}{x}$

f i $y = x(x-4)$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

$0 = x(x-4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

$y = -\frac{1}{x}$ is standard and in the second and fourth quadrants.

When $x = 2$,

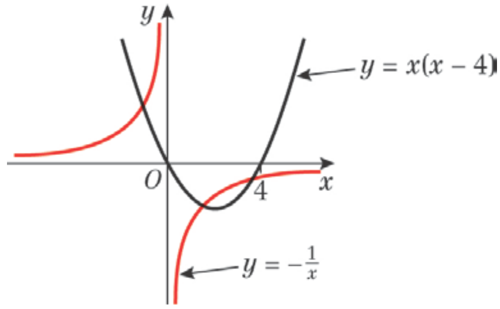
$y = -\frac{1}{x}$ gives $y = -\frac{1}{2}$

$y = x(x-4)$ gives $y = 2(-2) = -4$

So when $x = 2$, $x(x-4) < -\frac{1}{x}$

So $y = -\frac{1}{x}$ cuts $y = x(x-4)$ in the fourth quadrant.

f i



ii Three points of intersection

iii Equation: $x(x-4) = -\frac{1}{x}$

g i $y = x(x-4)$

As $a = 1$ is positive, the graph has a $\cup$ shape and a minimum point.

$0 = x(x-4)$

So $x = 0$ or $x = 4$

So the curve crosses the x-axis at (0, 0) and (4, 0).

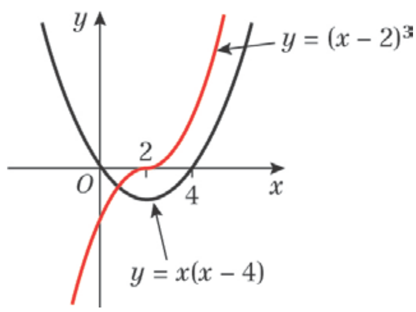
$y = (x-2)^3$

$0 = (x-2)^3$

So $x = 2$ and the curve crosses the x-axis at (2, 0) only.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$



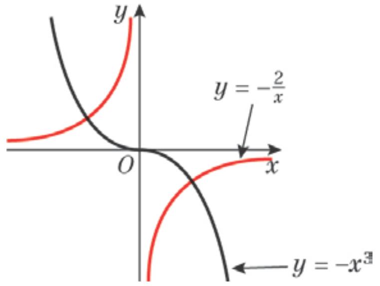
ii One point of intersection

iii $x(x-4) = (x-2)^3$

h i $y = -x^3$ is standard.

$y = -\frac{2}{x}$ is like $y = -\frac{1}{x}$ and in the second and fourth quadrants.

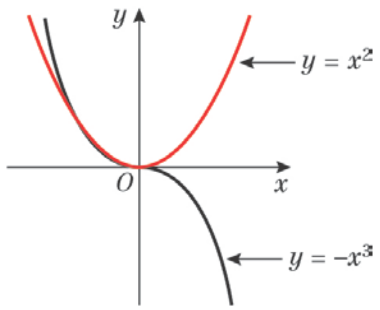
1 h i



ii Two points of intersection

iii $-x^3 = -\dfrac{2}{x}$ or $x^3 = \dfrac{2}{x}$

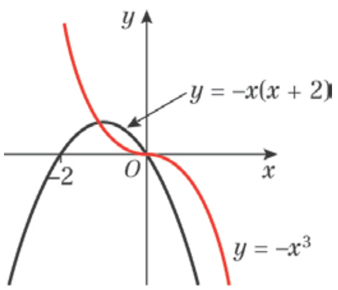
i i $y = -x^3$ is standard.
$y = x^2$ is standard.



ii Two points of intersection
(At (0, 0) the curves actually touch. They intersect in the second quadrant.)

iii $-x^3 = x^2$

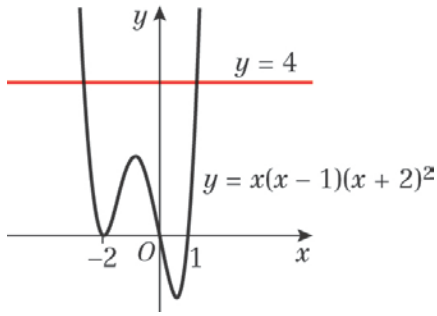
j i $y = -x^3$ is standard.
$y = -x(x + 2)$
As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point.
$0 = -x(x + 2)$
So $x = 0$ or $x = -2$
So the curve crosses the x-axis at (0, 0) and (−2, 0).



ii Three points of intersection

1 j iii $-x^3 = -x(x + 2)$ or $x^3 = x(x + 2)$

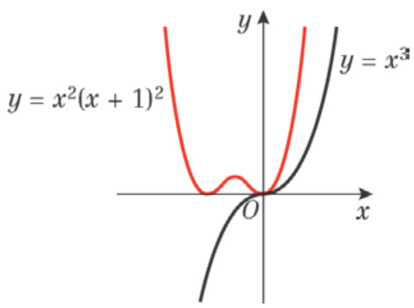
k i $y = 4$
$y = x(x - 1)(x + 2)^2$
$0 = x(x - 1)(x + 2)^2$
So $x = 0$, $x = 1$ or $x = -2$
The curve crosses the x-axis at (0, 0) and (1, 0) and touches it at (−2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$



ii Two points of intersection

iii $x(x - 1)(x + 2)^2 = 4$

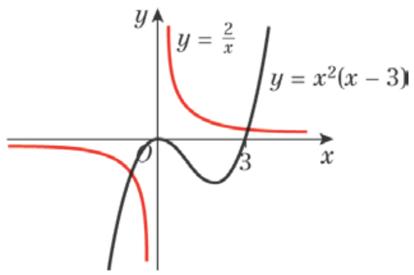
l i $y = x^3$ is standard.
$y = x^2(x + 1)^2$
$0 = x^2(x + 1)^2$
So $x = 0$ or $x = -1$
The curve touches the x-axis at (0, 0) and (−1, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$



ii One point of intersection

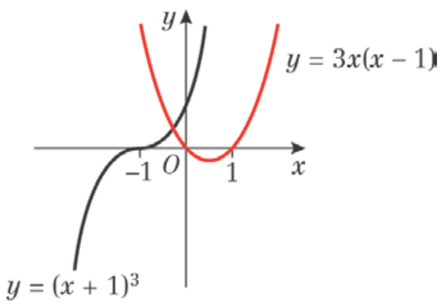
iii $x^3 = x^2(x + 1)^2$

2 a $y = x^2(x-3)$
$0 = x^2(x-3)$
So $x = 0$ or $x = 3$
The curve crosses the x-axis at (3, 0) and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = \frac{2}{x}$ is like $y = \frac{1}{x}$.



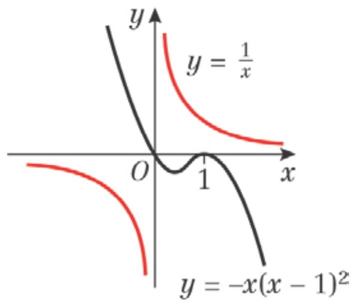
b From the sketch, there are only two points of intersection of the curves. This means there are only two values of x where
$x^2(x-3) = \frac{2}{x}$
$x^3(x-3) = 2$
So this equation has two real solutions.

3 a $y = (x+1)^3$
$0 = (x+1)^3$
So $x = -1$ and the curve crosses the x-axis at $(-1, 0)$ only.
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = 3x(x-1)$
As $a = 3$ is positive, the graph has a $\vee$ shape and a minimum point.
$0 = 3x(x-1)$
So $x = 0$ or $x = 1$
So the curve crosses the x-axis at (0, 0) and (1, 0).



b From the sketch, there is only one point of intersection of the curves. This means there is only one value of x where
$$(x+1)^3 = 3x(x-1)$$
$$x^3 + 3x^2 + 3x + 1 = 3x^2 - 1$$
$$x^3 + 6x + 1 = 0$$
So this equation has one real solution.

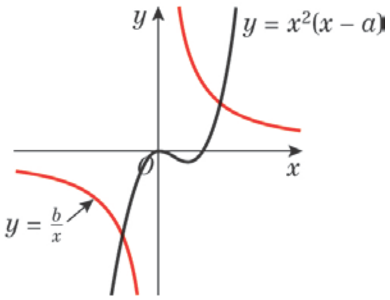
4 a $y = \frac{1}{x}$ is standard.
$y = -x(x-1)^2$
$0 = -x(x-1)^2$
So $x = 0$ or $x = 1$
The curve crosses the x-axis at (0, 0) and touches it at (1, 0).
$x \to \infty, y \to -\infty$
$x \to -\infty, y \to \infty$



b From the sketch, there are no points of intersection of the curves. This means there are no values of x where
$$\frac{1}{x} = -x(x-1)^2$$
$$1 = -x^2(x-1)^2$$
$$1 + x^2(x-1)^2 = 0$$
So this equation has no real solutions.

5 a $y = x^2(x-a)$
$0 = x^2(x-a)$
So $x = 0$ or $x = a$
The curve crosses the x-axis at $(a, 0)$ and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = \frac{b}{x}$ is a $y = \frac{k}{x}$ graph, with $k > 0$.

5 a



b From the sketch, there are two points of intersection of the curves. This means there are two values of x where

$$x^2(x-a)=\frac{b}{x}$$
$$x^3(x-a)=b$$
$$x^4-ax^3-b=0$$

So this equation has two real solutions.

6 a $y=\dfrac{4}{x^2}$ is a $y=\dfrac{k}{x^2}$ graph, with $k > 0$.

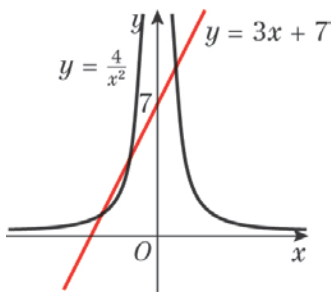
x^2 is always positive and $k > 0$ so the y-values are all positive.

$y = 3x + 7$

$0 = 3x + 7$

So $x=-\dfrac{7}{3}$

$y = 3x + 7$ is a straight line crossing the x-axis at $\left(-\frac{7}{3},0\right)$.



b There are three points of intersection, so there are three real solutions to the equation

$$\frac{4}{x^2}=3x+7$$

c $(x + 1)(x + 2)(3x - 2) = 0$

$$(x+1)(3x^2+4x-4)=0$$
$$3x^3+7x^2-4=0$$
$$3x^3+7x^2=4$$
$$x^2(3x+7)=4$$
$$3x+7=\frac{4}{x^2}$$

6 d $(x + 1)(x + 2)(3x - 2) = 0$

So $x = -1$, $x = -2$ or $x = \frac{2}{3}$

Using $y = 3x + 7$:

when $x = -1$, $y = 3(-1) + 7 = 4$

when $x = -2$, $y = 3(-2) + 7 = 1$

when $x = \frac{2}{3}$, $y = 3\left(\frac{2}{3}\right) + 7 = 9$

So the points of intersection are $(-1, 4)$, $(-2, 1)$ and $\left(\frac{2}{3},9\right)$.

7 a $y = x^3 - 3x^2 - 4x$

$= x(x^2 - 3x - 4)$

$0 = x(x - 4)(x + 1)$

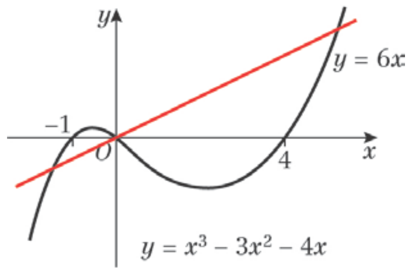
So $x = 0$, $x = 4$ or $x = -1$

The curve crosses the x-axis at $(0, 0)$, $(4, 0)$ and $(-1, 0)$.

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

$y = 6x$ is a straight line through $(0, 0)$.



b $x^3 - 3x^2 - 4x = 6x$

$x^3 - 3x^2 - 10x = 0$

$x(x^2 - 3x - 10) = 0$

$x(x - 5)(x + 2) = 0$

So $x = 0$, $x = 5$ or $x = -2$

Using $y = 6x$:

when $x = 0$, $y = 0$

when $x = 5$, $y = 30$

when $x = -2$, $y = -12$

So the points of intersection are $(0, 0)$, $(5, 30)$ and $(-2, -12)$.

8 a $y = (x^2 - 1)(x - 2)$

$= (x - 1)(x + 1)(x - 2)$

$0 = (x - 1)(x + 1)(x - 2)$

So $x = 1$, $x = -1$or $x = 2$

The curve crosses the x-axis at $(1, 0)$, $(-1, 0)$ and $(2, 0)$.

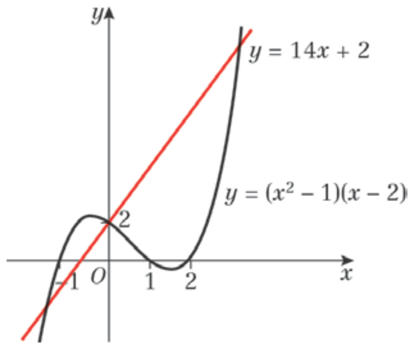
When $x = 0$, $y = (-1)^2 \times (-2) = 2$

$x \to \infty, y \to \infty$

$x \to -\infty, y \to -\infty$

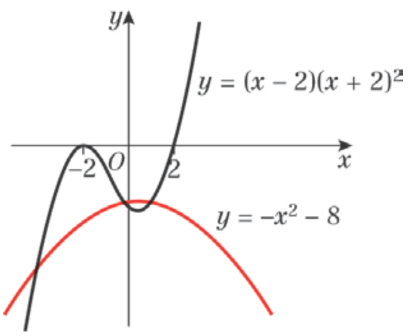
$y = 14x + 2$ is a straight line passing through $(0, 2)$ and $(-\frac{1}{7}, 0)$.

8 a



b $(x^2 - 1)(x - 2) = 14x + 2$
$x^3 - 2x^2 - x + 2) = 14x + 2$
$x^3 - 2x^2 - 15x = 0$
$x(x^2 - 2x - 15) = 0$
$x(x - 5)(x + 3) = 0$
$x = 0$, $x = 5$ or $x = -3$
Using $y = 14x + 2$:
when $x = 0$, $y = 2$
when $x = 5$, $y = 14(5) + 2 = 72$
when $x = -3$, $y = 14(-3) + 2 = -40$
So the points of intersection are (0, 2), (5, 72) and (−3, −40).

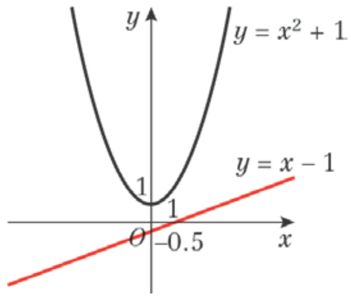
9 a $y = (x - 2)(x + 2)^2$
$0 = (x - 2)(x + 2)^2$
So $x = 2$ or $x = -2$
The curve crosses the x-axis at (2, 0) and touches it at (−2, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to -\infty$
$y = -x^2 - 8$
As $a = -1$ is negative, the graph has a $\cap$ shape and a maximum point at (0, −8).



b $(x + 2)^2(x - 2) = -x^2 - 8$
$(x^2 + 4x + 4)(x - 2) = -x^2 - 8$
$x^3 + 4x^2 + 4x - 2x^2 - 8x - 8 = -x^2 - 8$
$x^3 + 3x^2 - 4x = 0$
$x(x^2 + 3x - 4) = 0$
$x(x - 1)(x + 4) = 0$
So $x = 0$, $x = 1$ or $x = -4$

9 b Using $y = -x^2 - 8$:
when $x = 0$, $y = -0^2 - 8 = -8$
when $x = 1$, $y = -1^2 - 8 = -9$
when $x = -4$, $y = -(-4)^2 - 8 = -24$
So the points of intersection are (0, −8), (1, −9) and (−4, −24).

10 a $y = x^2 + 1$
As $a = 1$ is positive, the graph has a $\cap$ shape and a minimum point at (0, 1).
$2y = x - 1$
$y = \frac{1}{2}x - \frac{1}{2}$
This is a straight line passing through $\left(0, -\frac{1}{2}\right)$ and (1, 0).



b From the sketch, there are no points of intersection of the curves. This means there are no values of x where
$x^2 + 1 = \frac{1}{2}x - \frac{1}{2}$
$2x^2 + 2 = x - 1$
$2x^2 - x + 3 = 0$
So this equation has no real solutions.

c $x^2 + a = \frac{1}{2}x - \frac{1}{2}$
$2x^2 + 2a = x - 1$
$2x^2 - x + 2a + 1 = 0$
Using the discriminant for two real roots,
$b^2 - 4ac > 0$
$(-1)^2 - 4(2)(2a + 1) > 0$
$1 - 16a - 8 > 0$
$-16a - 7 > 0$
$16a < -7$
$a < -\frac{7}{16}$

11 a $y = x^2(x - 1)(x + 1)$
$0 = x^2(x - 1)(x + 1)$
So $x = 0$, $x = 1$ and $x = -1$
The curve crosses the x-axis at (1, 0) and (−1, 0) and touches it at (0, 0).
$x \to \infty, y \to \infty$
$x \to -\infty, y \to \infty$
$y = \frac{1}{3}x^3 + 1$

11 a $0 = \frac{1}{3}x^3 + 1$

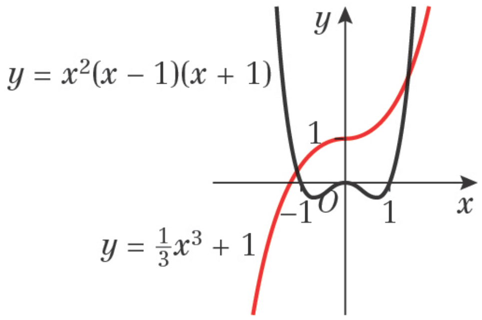
$\frac{1}{3}x^3 = -1$

$x^3 = -3$

$x = -\sqrt[3]{3}$

The curve crosses the x-axis at $\left(-\sqrt[3]{3}, 0\right)$.

When $x = 0$, $y = 1$



b From the sketch, there are two points of intersection of the curves. This means there are two values of x where

$x^2(x - 1)(x + 1) = \frac{1}{3}x^3 + 1$

$3x^2(x - 1)(x + 1) = x^3 + 3$

So this equation has two real solutions.